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Narciso Resende Gomes

Amostragem Compressiva em Análise de Clifford

Compressive Sensing in Clifford Analysis

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações (PDMA 20011-2015) da Universidade de Aveiro e da Universidade do Minho, realizada sob a orientação científica do Prof. Dr. Uwe Kähler, Professor Auxiliar c/ Agregação do Departamento de Matemática da Universidade de Aveiro e sob co-orientação científica da Prof.a Dr.a Paula Cerejeiras, Professora Associada do Departamento de Matemática da Universidade de Aveiro.

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presidente / president	**João Manuel Costa Araújo Pereira Coutinho** Professor Catedrático, Universidade de Aveiro, Portugal
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	Irene Sabadini Professora Catedrática, Politecnico di Milano, Itália
	Uwe Kähler Professor Auxiliar c/ Agregação, Universidade de Aveiro, Portugal (orientador)
	Qiuhui Chen Professor Catedrático, Guangdong University of Foreign Studies, Guangzhou, China
	Nelson Vieira Investigador Auxiliar, Universidade de Aveiro, Portugal

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> “*When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside and then noting the automobile road up the other side!*”
>
> Robert J. Kleinhenz,

Palavras-chave

Amostragem Compressiva; Análise Quaterniónica; Sinais Bicomplexos; Sinais Quaterniónicos; Sistema Takenaka-Malmquist; Esparsidade; Amostragem;

resumo

Amostragem Compressiva é um novo paradigma em processamento de sinal, no qual se assegura, para determinadas matrizes, que as representações esparsas de sinais podem ser obtidas por intermédio de um simples procedimento de ℓ_1-minimização. Nesta tese, exploramos este paradigma para sinais em dimensões superiores. Estudaremos três casos particulares: sinais com valores na álgebra bi-complexa, sinais quaterniónicos e, finalmente, sinais complexos representáveis por uma base de Fourier não-linear, dito sistema de Takenaka-Malmquist.

keywords Compressive Sensing; Quaternionic Analysis; Bicomplex signals; Quaternionic Signals; Takenaka-Malmquist system; Sparsity; Sampling.

abstract Compressed sensing is a new paradigm in signal processing which states that for certain matrices sparse representations can be obtained by a simple ℓ_1-minimization. In this thesis we explore this paradigm for higher-dimensional signal. In particular three cases are being studied: signals taking values in a bicomplex algebra, quaternionic signals, and complex signals which are representable by a nonlinear Fourier basis, a so-called Takenaka-Malmquist system.

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$\mathcal{L}^2(\mathbb{T})$	the Hilbert space of square integrable functions on the unit circle $\mathbb{T}$.
$\mathcal{H}^2(\mathbb{D})$	the Hardy space of complex functions analytic in $\mathbb{D}$.
$u : \Omega \to \mathbb{R}$ such that	
	$\|u\|_{L^p} = \left(\int_\Omega \|u\|^p \, dx\right)^{1/p} < \infty$, with $1 \leq p < \infty$
$L^\infty(\Omega)$	space of Lebesgue-measurable and essentially bounded functions
$L^2(\Omega)$	space of square-integrable functions on a continuous domain Ω
$\ell_0(\Omega)$	space of summable signals on a discrete domain Ω
$\ell_1(\Omega)$	space of summable signals on a discrete domain Ω
$\ell_2(\Omega)$	space of square-summable signals on a discrete domain Ω
$\langle \cdot, \cdot \rangle$	inner product (in a pre-Hilbert space)
$\|\cdot\|$	associated norm
$\|\cdot\|_p$	$p \geq 1, \ell_p$ norm of a signal
$\|\cdot\|_0$	ℓ_0 pseudo-norm of a signal; number of nonzero elements
$\|\cdot\|_F$	Frobenius norm of a matrix
$\mathbb{E}[\cdot]$	expectation operator
$\mathbf{M}^T$	transpose of a matrix $\mathbf{M}$
$\mathbf{M}^*$	adjoint of $\mathbf{M}$
$\mathbf{M}_{ij}$	entry at ith row and jth column of a matrix $\mathbf{M}$
$\mathrm{tr}(\mathbf{M})$	trace of a square matrix $\mathbf{M}$
$\mathrm{sgn}(x)$	signum function on variable x
I	identity operator or identity matrix of appropriate dimension;
$B_R(x)$	ball centered at x with radius R
$\int \cdot$	integral $\int_\Omega \cdot \, dx$
δ_{ij}	Kronecker deltas

Introduction

> *"Begin at the beginning," the King said, gravely, "and go on till you come to an end; then stop."*
>
> Lewis Carroll, *Alice's Adventures in Wonderland*, 1865

One of the principal problems in signal processing is the reconstruction of a signal from only a few samples. This is closely linked to other approximation problems like nonlinear and adaptive approximation. The main reason is that classic linear schemes in general cannot do the work. For instance, using the classic FFT (fast Fourier transform) one obtains N coefficients from N measurements(see, for instance [13]). While one can use greedy algorithms, i.e., in each step searching for the atom or basis element which fits best this does require exponentially growing computational costs and does not necessarily lead to exact reconstructions. In the last two decades one can observe a growing interest in this problem based on sparsity constraints. Sparsity constrains means that one imposes a-priori knowledge that the representation of the signal in a given basis or frame dictionary has only a small number of non-zero coefficients. This by itself would only mean that one can impose uniqueness in the representation of the signal in a basis which has to be still calculated by a greedy algorithm. The major achievement during these two decades consists in the development of schemes which replace the greedy algorithms in terms of linear and semi-linear scheme, such as the surrogate functional approach [27] or compressed sensing.

Compressed Sensing (CS) is a theory initially approached by Donoho [30] in 2006 and later by Candes, Romberg and Tao (CRT) [19] and so far applied in many fields of Applied Mathematics. The basic idea is that under certain conditions on the sampling matrix we can make a sparse recovery, i.e., make the acquisition and reconstruction of the high dimensional signals (vectors) exactly from a number of samples which is much smaller than the number of coefficients coming from non-adaptive linear projection observations, by means of a simple ℓ_1-minimization instead of using greedy methods. CS thus constitutes a paradigm shift in signal processing as well described in [47]. One of the problems of CS is that the condition on the sampling matrix (so-called RIP condition), i.e., for a given matrix A there exists a constant $\delta, 0 < \delta < 1$, such that

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2$$

for all s-sparse vectors x, is rather strong ([17]). Basically, it means that on any sparse vector the matrix behaves like an isometry. In practice this is difficult to fulfill. In many cases, particularly the ones coming from inverse problems ([79]), the matrices are two badly behaved for such a requirement. One principal way out was suggested by Rauhut [69] where the requirement is lowered to be fulfilled with a certain (high) probability. This is much more in line with application where worst-case scenarios are seldom or can entirely avoided. For instance, in case of finding an appropriate sampling matrix one could generate several to be sure that one of them fulfills the property. The work of Rauhut inspired the present work. Throughout the thesis we are interested to reconstruct a multivariate function with a few non-zero coefficients from random samples. Hereby, we propose to follow the idea of the reconstruction of the signal by basis pursuit, i.e., by ℓ_1-minimization [18, 22, 23].

This is also driven by the growing interest in using higher-dimensional algebraic structures in image processing which can represent the color image encoding during the last decade. In [65] the authors point out that the so-called IHS-color spaces representation (i.e., Intensity-Hue-Saturation) which has broad applications, particularly in human vision, can be mathematically represented by having values in bicomplex numbers. A color image using this coding scheme can thus be understood as a function from $\mathbb{R}^2$ into the bicomplex numbers. Other applications of these kind of bicomplex signals are in is multi-channel information [61].

Since an image can be understood as a sampled bicomplex signal this leads us to the question if it is possible, with high probability, to reconstruct a bicomplex signal with a few non-zero coefficients coming from a reduced number of bicomplex random samples. The advantage of using bicomplex numbers in order to obtain the result by compressed sensing techniques is the simple treatment by idempotent representation which makes the case similar to the case of the Fourier standard basis. This allows us to adapt arguments from the recent work of Rauhut [69] and CRT [19], where they study the problems of reconstruction of multivariate trigonometric polynomials having only few non-zero coefficients of random samples via ℓ_1-minimization (see [32]). Here, we will adapt his approach to obtain conditions on the sampling matrix which allows us to reconstruct a sparse signal by ℓ_1-minimization with high probability (we note that in this Thesis, for basic tools from probability theory, we consider [36], Chapter 7).

But the bicomplex signal, i.e., functions with values in the bicomplex algebra, is only the simplest example of the application of Clifford analysis to image processing (for instance, see [9, 20]). We hereby understood Clifford analysis ([72], [77]) in its broadest meaning, i.e., as a part of mathematical analysis where one studies a selected subset of functions, with domain in $\mathbb{R}^n$ and which take values in a particular hypercomplex algebra, with the prime example being a Clifford algebra. As such it generalizes the study of holomorphic functions in the plane to higher dimensions and can be understood as a generalization of complex analysis, although in current state of affairs it has outgrown this idea which can be seen that its modern methods differ quite a lot from classic complex analysis methods.

Since we are interested in applications to image processing we restrict ourselves to the simplest non-complex case, i.e., the case of quaternions. Due to the non-commutativity and the spatial freedom there exist a variety of possibilities to define a quaternionic Fourier transform and quaternionic Fourier atoms ([14, 33, 46, 66]). Although there is such a choice their mathematical theories in general are the same. Therefore, we only consider one case in this thesis, the case of Fourier atoms of the type $e^{\mathbf{I}kx}e^{\mathbf{J}ly}$ but our results are easily adaptable to the other cases. Applications of this Fourier transform can be found in ([8],[31, 58, 74, 80]) as well as in Bülow's hypercomplex signal ([14] and [16]), although in the later case we have in fact the two-sided Fourier transform. We did not consider applications to the monogenic signal, since we are mainly looking at the case of linear and non-linear Fourier atoms and we are not aware of any works which combines the monogenic signal with monogenic Gabor atoms. While a priori it is not clear that a quaternionic signal coming from an image must be sparse, to assume asymptotically sparse representations in his context is a natural assumption. Since quaternionic signals can be considered as boundary values of analytic (in one context or another) functions they will have a fast enough decay in a series expansion to be "sufficiently sparse" in a numerical context if the series expansion is reasonably chosen.

As a third part we are interested in a generalization of the standard Fourier basis, the case of so-called nonlinear Fourier atoms. Decomposition algorithms for this kind of atoms were thoroughly investigated by the group of T. Qian in Macau during the last decade ([25]). Although the original starting point for T. Qian was the investigation into a mathematical justification of the Hilbert-Huang transform and the empirical mode decomposition, the underlying structure is much older. The whole approach is in fact based on the question of decomposing a function on the unit circle in terms of Blaschke products, that means in terms of the so-called Takenaka-Malmquist system. Although being one of the classic topics in Complex Analysis this system is rather unknown in the signal processing community with its focus on Wavelet and Gabor decompositions (see, for instance [55]). That is the main reason why investigations from the point of signal processing into this system basically restarted within the last decade, not only by T. Qian, but also by M. Pap and her collaborators in the framework of the study of the so-called Voice transform ([63] and [64]). Due to its close connection with the group of Möbius transformations they can also be used to describe dilated functions on the unit circle which was used in the definition of hyperbolic wavelets. While the adaptive Fourier decompositions of T. Qian showed its capacities in a variety of examples, principally, in linear systems theory, it still constitutes a greedy algorithm and it is not clear that in applications the number of atoms can be kept sufficiently small as not to be affected by the exponentially rising costs. In fact in the recent PhD-thesis by L. Shuang (c.f. [75]), a comparison between the AFD (Adaptive Fourier Decomposition)-method and Basis Pursuit where made, although the mathematical justification for the applicability of Basis Pursuit was given only by an asymptotic analysis and, therefore, only valid for large scale matrices. Here we will use the approach given for the bi-complex and quaternionic signals to give a general answer. But this represents an additional problem. The calculation of the expectation value turns out to be much more demanding due to the lack of structure (no easy multiplication rule).

This thesis is organized as follows:

In Chapter 1, we study the Compressed Sensing methods for the bicomplex setting. For this purpose we apply the idempotent decomposition from the algebra which allows us treat the case quite similar to the case of classic Fourier atoms.

In Chapter 2, we study the quaternionic signals. Indeed this is the part where we have more difficulties. The quaternionic numbers, as we know, is non-commutative. This property makes it harder when we have to study powers of quaternionic matrices.

In Chapter 3, we consider the Takenaka-Malmquist system coming from a non-linear Fourier basis.

In Chapter 4, we present some final considerations on the three studied problems and give some direction on possible future research.

Chapter 1

Bicomplex signal

> “ *For the wise man looks into space and he knows there is no limited dimensions.* ”
>
> Lao Tzu,

In this chapter we aim to proof the possibility to reconstruct a bicomplex sparse signal, with high probability, from a reduced number of bicomplex random samples. Due to the idempotent representation of the bicomplex algebra this case is similar to the case of the standard Fourier basis, thus allowing us to adapt in a rather easy way the arguments from the recent works of Rauhut [69] and Candés et al. [19].

1.1 Bicomplex Analysis

In the following subsections we introduce the hyperbolic and the bicomplex algebras, we present different automorphisms in these algebras, possible norms, discuss their decomposition into ideals based on their zero divisors, before passing to the description of a function theory over these spaces.

1.1.1 The Hyperbolic and Bicomplex Algebras

The two most common extensions of the real algebra $\mathbb{R}$ are the complex algebra $\mathbb{C}$ and the quaternionic algebra $\mathbb{H}$ (respectively, dimensions 2 and 4). Also, it is well-known that an increase in dimension is made at the expenses of properties of the algebra. Thus, the complex algebra is still a division algebra but no longer possesses an order relation, while the quaternionic algebra is still a division algebra but is not a commutative algebra anymore[3, 4, 37, 44].

However, these are not the only possible realizations of 2 and 4 dimensional real algebras. For the $2D$ case, we can define also the set of hyperbolic numbers, denote by $\mathbb{D}$, and defined as

$$\mathbb{D} := \{a + b\mathbf{k} \,|\, a, b \in \mathbb{R},\, \mathbf{k}^2 = 1\} \quad (\mathbf{k} \notin \mathbb{R}). \tag{1.1}$$

Endowed with the usual addition and the multiplication

$$(a + b\mathbf{k})(c + d\mathbf{k}) = (ac + bd) + (ad + bc)\mathbf{k},$$

the space (1.1) becomes a real algebra, isomorphic to $\mathbb{R}^2$, and but it is not a division algebra. This algebra is associated to the quadratic form

$$Q(a + b\mathbf{k}) := a^2 - b^2.$$

The quadratic form divides the space (1.1) into two regions, one of which is connected, namely, the region associated to $Q(a + b\mathbf{k}) = a^2 - b^2 \leq 0$, while the second, $Q(a + b\mathbf{k}) = a^2 - b^2 \geq 0$, is a non-connected domain. For practical proposes we shall divided this last into two subdomains. We denote as $\mathbb{D}^+$ the subdomain defined by $Q(a + b\mathbf{k}) = a^2 - b^2 \geq 0$ and $a \geq 0$, or *domain of positive hyperbolic numbers*. We represent by $\mathbb{D}^-$ the *domain of negative hyperbolic numbers*, that is, defined by $Q(a + b\mathbf{k}) = a^2 - b^2 \geq 0$ and $a < 0$.

As a realization of a commutative $4D$ real algebra (again not a division algebra) we have the commutative ring of bicomplex numbers, denoted as $\mathbb{BC}$. Here, $\mathbf{i}$ and $\mathbf{j}$ are commuting imaginary units, i.e.,

$$\mathbf{ij} = \mathbf{ji} = \mathbf{k}, \quad \mathbf{i}^2 = \mathbf{j}^2 = -1,$$

and

$$\mathbb{BC} := \{\zeta = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, a, b, c, d \in \mathbb{R}\}.$$

Based on $\mathbf{ij} = \mathbf{ji} = \mathbf{k}$ we can decompose the elements of $\mathbb{BC}$ into two ways. First,

$$\zeta = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = z_1 + z_2\mathbf{j}, \quad z_1 = a + b\mathbf{i}, z_2 = c + d\mathbf{i} \in \mathbb{C}(\mathbf{i}),$$

where $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with the imaginary unit $\mathbf{i}$. Secondly,

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \eta_1 + \eta_2\mathbf{i}, \quad \eta_1 = a + c\mathbf{j}, \eta_2 = b + d\mathbf{j} \in \mathbb{C}(\mathbf{j}),$$

where $\mathbb{C}(\mathbf{j})$ is the set of complex numbers with the imaginary unit $\mathbf{j}$.

Also, we remark that

$$\mathbb{R}^4 \ni (x_1, y_1, x_2, y_2) \sim \zeta = x_1 + y_1\mathbf{i} + x_2\mathbf{j} + y_2\mathbf{k} \in \mathbb{BC}. \tag{1.2}$$

Hence, topological concepts like a subset of $\mathbb{BC}$ being *open, connected,* etc. are to be understood via this identification (1.2), with the usual Euclidean norm. Moreover, we can embed the set of hyperbolic numbers $\mathbb{D}$ into $\mathbb{BC}$ as follows.

$$z = a + b\mathbf{k} \quad \sim \quad a + 0\mathbf{i} + 0\mathbf{j} + b\mathbf{ij} \in \mathbb{BC}.$$

1.1.2 Automorphism and the Euclidean Norm on $\mathbb{BC}$

Since $\mathbb{BC}$ contains two imaginary units, the elements $\mathbf{i}$ and $\mathbf{j}$, and one hyperbolic unit, the element $\mathbf{k}$, we can consider three possible automorphisms, or conjugations, for bicomplex numbers. Each one can be regarded as an analogue to the usual complex conjugation.

(a) $\zeta := z_1 + z_2\mathbf{j} \mapsto \overline{\zeta} := \overline{z}_1 + \overline{z}_2\mathbf{j}$ (or bar-conjugation);

(b) $\zeta := z_1 + z_2\mathbf{j} \mapsto \zeta^\dagger := z_1 - z_2\mathbf{j}$ (or †-conjugation);

(c) $\zeta := z_1 + z_2\mathbf{j} \mapsto \zeta^* := \left(\overline{\zeta}\right)^\dagger = \overline{(\zeta^\dagger)} = \overline{z}_1 - \overline{z}_2\mathbf{j}$ (or $*$-conjugation).

In the complex case the product of a complex number with its conjugate defines a quadratic form which coincides with the square of its modulus, or its Euclidean norm when viewed as a vector of $\mathbb{R}^2$. For the bicomplex case, and based on the three possible conjugations, we have

(a) $|\zeta|_{\mathbf{j}}^2 := \zeta\overline{\zeta} = |z_1|^2 - |z_2|^2 + 2\mathrm{Re}\left(z_1\overline{z_2}\right)\mathbf{j} \in \mathbb{C}(\mathbf{j})$;

(b) $|\zeta|_{\mathbf{i}}^2 := \zeta\zeta^\dagger = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i})$;

(c) $|\zeta|_{\mathbf{k}}^2 := \zeta\zeta^* = |z_1|^2 + |z_2|^2 - 2\mathrm{Im}\left(z_1\overline{z_2}\right)\mathbf{k} \in \mathbb{D}$.

Unlike the complex case, these moduli are not $\mathbb{R}_0^+$-valued. The first two moduli are complex-valued ($\mathbb{C}(\mathbf{j})$ and $\mathbb{C}(\mathbf{i})$, respectively), while the last one is $\mathbb{D}-$valued.

Since $\mathbb{BC}$ can be identified with $\mathbb{R}^4$, we wish to express the usual Euclidean norm on $\mathbb{R}^4$ in terms of the previous decompositions. For that, we define the following spaces.

1. $\mathbb{C}^2(\mathbf{i}) = \{(z_1, z_2) \,|\, z_1, z_2 \in \mathbb{C}(\mathbf{i})\} \cong \{z_1 + z_2\mathbf{j} \in \mathbb{BC} \,|\, z_1, z_2 \in \mathbb{C}(\mathbf{i})\} = \mathbb{BC}$;

2. $\mathbb{C}^2(\mathbf{j}) = \{(\eta_1, \eta_2) \,|\, \eta_1, \eta_2 \in \mathbb{C}(\mathbf{j})\} \cong \{\eta_1 + \eta_2\mathbf{i} \in \mathbb{BC} \,|\, \eta_1, \eta_2 \in \mathbb{C}(\mathbf{j})\} = \mathbb{BC}$.

The Euclidean norm $|(x_1, y_1), (x_2, y_2)| = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}$ can be written in terms of the elements of $\mathbb{C}^2(\mathbf{i})$ and of $\mathbb{C}^2(\mathbf{j})$ via the $\mathbb{D}^+$-valued modulus as follows.

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = |z_1|^2 + |z_2|^2 = |\eta_1|^2 + |\eta_2|^2 = \mathrm{Re}\left(|\zeta|_{\mathbf{k}}^2\right), \tag{1.3}$$

and, for every $\zeta, \xi \in \mathbb{BC}$ it holds (by the triangle inequality),

$$|\zeta\xi| \le \sqrt{2}|\zeta|\,|\xi|.$$

Also, for an arbitrary $\zeta = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$, with $z_1, z_2 \in \mathbb{C}(\mathbf{i})$, we have

1. if $\xi \in \mathbb{C}(\mathbf{i})$ then $|\zeta\xi| \le |\zeta|\,|\xi|$. The same holds if $\xi \in \mathbb{C}(\mathbf{j})$;

2. if $\xi \in \mathbb{D}$ then,
$$|\zeta\xi|^2 = |\zeta|^2|\xi|^2 + 4xy\mathrm{Re}(\mathbf{i}z_1\overline{z}_2)$$
where $\xi = x + \mathbf{k}y$.

1.1.3 Idempotent Decompositions

As seen in the previous subsection for an arbitrary bicomplex $\zeta = z_1 + z_2\mathbf{j}$ it holds

$$|\zeta|_{\mathbf{i}}^2 = \zeta\zeta^\dagger = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i}).$$

Then, it follows that every bicomplex number ζ satisfying $|\zeta|_{\mathbf{i}} \neq 0$ is invertible, and its inverse is given by

$$\zeta^{-1} = \frac{\zeta^\dagger}{|\zeta|_{\mathbf{i}}^2}.$$

If, on the other hand, $|\zeta|_{\mathbf{i}} = 0$ then ζ is a zero divisor in $\mathbb{BC}$. In fact, there are no other zero divisors in the algebra. We shall denote the set of all zero divisors in $\mathbb{BC}$ by $\mathfrak{S}$. Thus,

$$\mathfrak{S} := \{\zeta = z_1 + z_2\mathbf{j} \in \mathbb{BC} \mid z_1^2 + z_2^2 = 0\}.$$

We now turn our attention to two special zero divisors. We define

$$\mathbf{e}^+ = \frac{1}{2}(1 + \mathbf{ij})$$

and its †-conjugate

$$\mathbf{e}^- = \frac{1}{2}(1 - \mathbf{ij}).$$

Immediate computations give

$$\mathbf{e}^+\mathbf{e}^- = \mathbf{e}^-\mathbf{e}^+ = 0.$$

This means that $\mathbf{e}^+$ and $\mathbf{e}^-$ are zero divisors of the algebra. Moreover, they are mutually complementary idempotent elements since

$$\mathbf{e}^+ + \mathbf{e}^- = 1, \quad (\mathbf{e}^+)^2 = \mathbf{e}^+, \quad (\mathbf{e}^-)^2 = \mathbf{e}^-.$$

Thus, the two sets

$$\mathbb{BC}_{\mathbf{e}^+} := \mathbb{BC}\mathbf{e}^+ \quad \text{and} \quad \mathbb{BC}_{\mathbf{e}^-} := \mathbb{BC}\mathbf{e}^- \tag{1.4}$$

are principal ideals in the ring $\mathbb{BC}$, with

$$\mathbb{BC} = \mathbb{BC}_{\mathbf{e}^+} + \mathbb{BC}_{\mathbf{e}^-}, \quad \mathbb{BC}_{\mathbf{e}^+} \cap \mathbb{BC}_{\mathbf{e}^-} = \{0\}. \tag{1.5}$$

Moreover, given an open set $X \subset \mathbb{BC}$, we shall denote its *projections* into the principal ideals $\mathbb{BC}_{\mathbf{e}^+}$ and $\mathbb{BC}_{\mathbf{e}^-}$ as

$$X_1 = \Pi_{\mathbf{e}^+}X \subset \mathbb{BC}_{\mathbf{e}^+}, \quad X_2 = \Pi_{\mathbf{e}^-}X \subset \mathbb{BC}_{\mathbf{e}^-}.$$

Taking into account (1.4) and (1.5) we have $X = X_1\mathbf{e}^+ + X_2\mathbf{e}^-$, with $X_1\mathbf{e}^+ = X\mathbf{e}^+$ and $X_2\mathbf{e}^- = X\mathbf{e}^-$.

We shall call (1.5) the idempotent decomposition of $\mathbb{BC}$. Of course, both ideals, $\mathbb{BC}_{\mathbf{e}^+}$ and

$\mathbb{BC}_{\mathbf{e}^-}$, are uniquely determined but their elements admit different representations. In fact, every bicomplex number $\zeta = z_1 + z_2\mathbf{j} \in \mathbb{C}^2(\mathbf{i})$ can be written as

$$\zeta = \beta_1\mathbf{e}^+ + \beta_2\mathbf{e}^-, \quad \text{where } \beta_1 = z_1 - \mathbf{i}z_2,\ \beta_2 = z_1 + \mathbf{i}z_2 \in \mathbb{C}(\mathbf{i}), \tag{1.6}$$

or alternatively, if $\zeta = \eta_1 + \eta_2\mathbf{i} \in \mathbb{C}^2(\mathbf{j})$, as

$$\zeta = \gamma_1\mathbf{e}^+ + \gamma_2\mathbf{e}^-, \quad \text{where } \gamma_1 = \eta_1 - \mathbf{j}\eta_2,\ \gamma_2 = \eta_1 + \mathbf{j}\eta_2 \in \mathbb{C}(\mathbf{j}). \tag{1.7}$$

Of importance for us is the relation between the idempotent representations (1.6) and (1.7). Indeed, this representation leads to the so-called *hyperbolic norm*(see [2])

$$|\zeta|_{\mathbf{k}} = |\beta_1|\mathbf{e}^+ + |\beta_2|\mathbf{e}^- = |\gamma_1|\mathbf{e}^+ + |\gamma_2|\mathbf{e}^-. \tag{1.8}$$

Let us point out that (1.8) is, of course, not a norm. It gives rise to the following norm

$$2\mathrm{Re}(|\zeta|_{\mathbf{k}}) = 2\mathrm{Re}(|\beta_1|\mathbf{e}^+ + |\beta_2|\mathbf{e}^-) := |\beta_1| + |\beta_2|, \tag{1.9}$$

which stands for a norm equivalent to the Euclidean norm (1.3. Nevertheless, keeping in mind that the norm (1.9) in many cases is easier to work with than (1.8), as can be seen in the sequel.

A similar decomposition holds for hyperbolic numbers. First of all, recall that $\mathbf{k} = \mathbf{ij}$, so that both $\mathbf{e}^+$ and $\mathbf{e}^-$ can be viewed as hyperbolic numbers. Hence, the idempotent representation of a hyperbolic number $\alpha = a + b\mathbf{k}$ is

$$\alpha = \nu\mathbf{e}^+ + \mu\mathbf{e}^-,$$

with $\nu = b + a, \mu = b - a \in \mathbb{R}$. As $Q(\alpha) = a^2 - b^2 = \nu\mu$ we have that

$$\mathbb{D}^+ = \{\nu\mathbf{e}^+ + \mu\mathbf{e}^- \mid \nu, \mu \in \mathbb{R}_0^+\}.$$

Thus, positive hyperbolic numbers are those whose both idempotent components are nonnegative.

We now look into idempotent representations of matrices taking values in $\mathbb{BC}$. We denote by $\mathbb{BC}^{m\times n}$ the set of $m \times n$ matrices with bicomplex entries. As in the scalar case, the operations over the matrices can be realized component-wise keeping in mind the non-commutativity of matrix multiplication. Note that $\mathbb{BC}^{m\times n}$ is not a vectorial space, but a $\mathbb{BC}$-module.

Given such a matrix $A = (a_{i,j}) \in \mathbb{BC}^{m\times n}$ its idempotent representation is obtained by accordingly decomposing each of its entries by (1.6), that is,

$$A = A_1\mathbf{e}^+ + A_2\mathbf{e}^- \tag{1.10}$$

where $A_1 = \Pi_{\mathbf{e}^+}A, A_2 = \Pi_{\mathbf{e}^-}A$ are now $m \times n$ matrices taking values in $\mathbb{C}(\mathbf{i})$. In consequence, we shall write $A_1, A_2 \in \mathbb{C}^{m\times n}(\mathbf{i})$, and we have

$$\mathbb{BC}^{m\times n} = \mathbb{C}^{m\times n}(\mathbf{i})\mathbf{e}^+ + \mathbb{C}^{m\times n}(\mathbf{i})\mathbf{e}^-.$$

Definition 1.1.1. *Given a square matrix* $A = A_1\mathbf{e}^+ + A_2\mathbf{e}^- \in \mathbb{C}^{n\times n}(\mathbf{i})\mathbf{e}^+ + \mathbb{C}^{n\times n}(\mathbf{i})\mathbf{e}^- = \mathbb{BC}^{n\times n}$, *we define its determinant* $\det A$ *by*

$$\det A = (\det A_1)\mathbf{e}^+ + (\det A_2)\mathbf{e}^-.$$

Lemma 1.1.2 ([2], Corollary 2.2.2)**.** *Let be* A *and* B *be two square bicomplex matrices. Then*

$$\det(AB) = \det A \det B.$$

Lemma 1.1.3 ([2], Proposition 2.2.3)**.** *A square bicomplex matrix* $A = A_1\mathbf{e}^+ + A_2\mathbf{e}^- \in \mathbb{C}^{n\times n}(\mathbf{i})\mathbf{e}^+ + \mathbb{C}^{n\times n}(\mathbf{i})\mathbf{e}^- = \mathbb{BC}^{n\times n}$ *is invertible if and only if* A_1, A_2 *are invertible in* $\mathbb{C}^{n\times n}(\mathbf{i})$.

1.1.4 Bicomplex holomorphic functions

In order to compute the expectation value of a bicomplex random variable we require a type of Cauchy's integral formula for functions taking values in $\mathbb{BC}$. With that end in view, we start with the appropriated concept of derivative of a bicomplex function and proceed with the Cauchy's integral formula based on the previous idempotent representations established in Subsection 1.1.3. Proofs of these results shall be omitted as they can be found in [5]. For the interested reader, we recommend also [2].

Let X be a non-empty and connected domain of $\mathbb{BC}$. A function $f : X \subset \mathbb{BC} \cong \mathbb{C}^2(\mathbf{i}) \to \mathbb{BC}$, where

$$\zeta \mapsto f(\zeta) = u(\zeta) + v(\zeta)\mathbf{j}, \quad (u, v : X \subset \mathbb{C}^2(\mathbf{i}) \to \mathbb{C}(\mathbf{i})),$$

is continuous in X if all of its components are continuous there (in the classic real-valued sense).

Definition 1.1.4 ([5], Definition 20.13)**.** *Let* X *be a non-empty and connected domain of* $\mathbb{BC}$. *A function* $f : X \subset \mathbb{BC} \cong \mathbb{C}^2(\mathbf{i}) \to \mathbb{BC}$ *as a derivative* $f'(\zeta_0)$ *at* $\zeta_0 \in X \subset \mathbb{BC} \cong \mathbb{C}^2(\mathbf{i})$ *if and only if the limit*

$$f'(\zeta_0) := \lim_{\mathfrak{S}_0 \not\ni \mathfrak{h} \to 0} \frac{f(\zeta_0 + \mathfrak{h}) - f(\zeta_0)}{\mathfrak{h}}, \tag{1.11}$$

exists, where $\mathfrak{h} = h_1 + \mathbf{j}h_2 \in \mathbb{C}^2(\mathbf{i})$ *is an invertible bicomplex number. In this case, we say that* f *is differentiable at* ζ_0.

If f has a bicomplex derivative at each point of $X \subset \mathbb{C}^2(\mathbf{i})$, we say that f is a *bicomplex holomorphic* function in X. We now proceed in analogy with holomorphic functions in one complex variable. More explicitly, if the bicomplex function $f = u + \mathbf{j}v$ has a derivative at ζ_0, then its complex partial derivatives

$$f'_{z_1}(\zeta_0) := \lim_{h_1 \to 0} \frac{f(\zeta_0 + h_1) - f(\zeta_0)}{h_1}, \quad f'_{z_2}(\zeta_0) := \lim_{h_2 \to 0} \frac{f(\zeta_0 + \mathbf{j}h_2) - f(\zeta_0)}{h_2},$$

exist and verify the identity $f'(\zeta_0) = f'_{z_1}(\zeta_0) = -\mathbf{j}f'_{z_2}(\zeta_0)$, there is to say, its components satisfy the *complex Cauchy-Riemann system*

$$\begin{cases} \frac{\partial u}{\partial z_1}(\zeta_0) = \frac{\partial v}{\partial z_2}(\zeta_0) \\ \\ \frac{\partial u}{\partial z_2}(\zeta_0) = -\frac{\partial v}{\partial z_1}(\zeta_0). \end{cases} \tag{1.12}$$

Conversely, if $u, v : X \cong \mathbb{C}^2(\mathbf{i}) \to \mathbb{C}(\mathbf{i})$ are differentiable functions (in the several complex variables sense) and satisfy (1.12) in X, then $f = u + v\mathbf{j}$ is bicomplex holomorphic in X (see [5], Theorem 24.2). Obviously, dual conditions hold for X realized as a subset of $\mathbb{C}^2(\mathbf{j})$.

Of particular interest for us is the decomposition of a bicomplex holomorphic function in terms of its projections into the principal ideals $\mathbb{BC}_{\mathbf{e}^+}$ and $\mathbb{BC}_{\mathbf{e}^-}$.

Theorem 1.1.5 ([5], Theorems 21.1 and 24.3)**.** *Let $X \in \mathbb{C}^2(\mathbf{i})$ be a domain, and take its projections $X_1 = \Pi_{\mathbf{e}^+}X$ and $X_2 = \Pi_{\mathbf{e}^-}X$ in $\mathbb{C}(\mathbf{i})$. Let $f_1 : X_1 \subset \mathbb{C}(\mathbf{i}) \to \mathbb{C}(\mathbf{i}),\ z_1 - \mathbf{i}z_2 \mapsto f_1(z_1 - \mathbf{i}z_2)$, and $f_2 : X_2 \subset \mathbb{C}(\mathbf{i}) \to \mathbb{C}(\mathbf{i}),\ z_1 + \mathbf{i}z_2 \mapsto f_2(z_1 + \mathbf{i}z_2)$, be complex-valued functions. Then, the function*

$$f(\zeta) = f_1\left(z_1 - \mathbf{i}z_2\right)\mathbf{e}^+ + f_2\left(z_1 + \mathbf{i}z_2\right)\mathbf{e}^- \in \mathbb{C}^2(\mathbf{i}), \tag{1.13}$$

where $\zeta = z_1 + \mathbf{j}z_2 \in X = X_1 + X_2$, is a bicomplex holomorphic function if and only if f_1 and f_2 are holomorphic functions with respect to $z_1 - \mathbf{i}z_2$ and $z_1 + \mathbf{i}z_2$, respectively.

Moreover,

$$f'(\zeta) = \partial_{z_1 - \mathbf{i}z_2} f_1(z_1 - \mathbf{i}z_2)\mathbf{e}^+ + \partial_{z_1 + \mathbf{i}z_2} f_2(z_1 + \mathbf{i}z_2)\mathbf{e}^-.$$

Before stating Cauchy's integral formula for bicomplex holomorphic functions, we first define a *curve* in $\mathbb{BC} \cong \mathbb{C}^2(\mathbf{i})$ as a mapping (with continuous derivative) $\zeta(\cdot) : [a, b] \subset \mathbb{R} \to \mathbb{C}^2(\mathbf{i})$,

$$t \mapsto \zeta(t) = z_1(t) + \mathbf{j}z_2(t) := \left(z_1(t) - \mathbf{i}z_2(t)\right) + \mathbf{j}\left(z_1(t) + \mathbf{i}z_2(t)\right). \tag{1.14}$$

Obviously, each such curve admits an idempotent decomposition

$$t \mapsto \zeta(t) = \zeta_1(t)\mathbf{e}^+ + \zeta_2(t)\mathbf{e}^- = \left(z_1(t) - \mathbf{i}z_2(t)\right)\mathbf{e}^+ + \left(z_1(t) + \mathbf{i}z_2(t)\right)\mathbf{e}^-,$$

where ζ_1, ζ_2 define two (continuous and differentiable) curves in $\mathbb{C}(\mathbf{i})$.

Also, given a point $a = a_1 + \mathbf{j}a_2 \in \mathbb{C}^2(\mathbf{i})$, and $r_1, r_2 > 0$, we define the closed disc $\bar{D}(a; r_1, r_2)$ with center a and radius r_1 and r_2 as follows

$$\bar{D}(a; r_1, r_2) = \{z_1 + \mathbf{j}z_2 \in \mathbb{C}^2(\mathbf{i})\ :\ |(z_1 - \mathbf{i}z_2) - (a_1 - \mathbf{i}a_2)| \le r_1\ \&\ |(z_1 + \mathbf{i}z_2) - (a_1 + \mathbf{i}a_2)| \le r_2\}.$$

We state our Cauchy's integral formula.

Theorem 1.1.6 (Cauchy Integral Formula, c.f. [5], Theorem 40.3)**.** *Let $f : X \subset \mathbb{C}^2(\mathbf{i}) \to \mathbb{C}^2(\mathbf{i})$ be a bicomplex holomorphic function, consider the disc $\bar{D}(a; r_1, r_2)$ define above, and let C be the* *curve*

$$t \mapsto z_1(t) + \mathbf{j}z_2(t) := (a_1(t) + \mathbf{j}a_2(t)) + (r_1\mathbf{e}^+ + r_2\mathbf{e}^-)(\cos t + \mathbf{i}\sin t), \quad t \in [0, 2\pi].$$

Then, for every $w_1 + \mathbf{j}w_2 \in \bar{D}(a; r_1, r_2)$ *it holds*

$$f(w_1 + \mathbf{j}w_2) = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(\zeta)d\zeta}{\zeta - (w_1 + \mathbf{j}w_2)}. \tag{1.15}$$

Moreover, if f *admits the idempotent decomposition* $f(z_1+\mathbf{j}z_2) = f_1(z_1 - \mathbf{i}z_2)\,\mathbf{e}^+ + f_2(z_1 + \mathbf{i}z_2)\,\mathbf{e}^-$ *and the curves* $C_1 = \Pi_{\mathbf{e}^+}C,\ C_2 = \Pi_{\mathbf{e}^-}C$ *are positively oriented curves in* $\mathbb{C}(\mathbf{i})$*, then,*

$$\begin{aligned} f(w_1 + \mathbf{j}w_2) &= f_1(w_1 - \mathbf{i}w_2)\,\mathbf{e}^+ + f_2(w_1 + \mathbf{i}w_2)\,\mathbf{e}^- \\ &= \frac{1}{2\pi\mathbf{i}} \int_{C_1} \frac{f_1(\zeta_1)d\zeta_1}{\zeta_1 - (w_1 - \mathbf{i}w_2)}\mathbf{e}^+ + \frac{1}{2\pi\mathbf{i}} \int_{C_2} \frac{f_2(\zeta_2)d\zeta_2}{\zeta_2 - (w_1 + \mathbf{i}w_2)}\mathbf{e}^-, \end{aligned} \tag{1.16}$$

where $\zeta_1(t) = z_1(t) - \mathbf{i}z_2(t),\ \zeta_2(t) = z_1(t) + \mathbf{i}z_2(t),$ *for* $t \in [0, 2\pi]$.

1.2 Sparse sampling of Bicomplex signals

We now consider the problem of reconstructing a sparse bicomplex signal. We shall begin with a description of our setting, after which we present our main results.

1.2.1 The Bicomplex Setting

We consider bicomplex trigonometric waves of type

$$\zeta_{k,\tilde{k}}(x, y) = e^{\mathbf{i}kx}\mathbf{e}^+ + e^{\mathbf{i}\tilde{k}y}\mathbf{e}^-, \quad x, y \in [0, 2\pi], \tag{1.17}$$

with $k, \tilde{k} \in \mathbb{Z}$. As in [69], we denote by $\prod_\rho$ the space of all bicomplex trigonometric polynomials of maximal order $\rho \in \mathbb{N}_0$ in two real variables. For convenience, we use $\mathbb{Z}^2_\rho = [-\rho, \rho]^2 \cap \mathbb{Z}^2$. Thus, an element $f \in \prod_\rho$ is of the form

$$\begin{aligned} f(x, y) &= \sum_{(k,\tilde{k}) \in \mathbb{Z}^2_\rho} c_{k,\tilde{k}} \left(e^{\mathbf{i}kx}\mathbf{e}^+ + e^{\mathbf{i}\tilde{k}y}\mathbf{e}^- \right), \\ &= \sum_{(k,\tilde{k}) \in \mathbb{Z}^2_\rho} \left(c_k e^{\mathbf{i}kx}\mathbf{e}^+ + c_{\tilde{k}} e^{\mathbf{i}\tilde{k}y}\mathbf{e}^- \right), \qquad (x, y) \in [0, 2\pi]^2, \end{aligned} \tag{1.18}$$

for some sequence of coefficients $c = (c_{k,\tilde{k}})_{k,\tilde{k}}$, where

$$c_{k,\tilde{k}} = c_k\mathbf{e}^+ + c_{\tilde{k}}\mathbf{e}^- \in \mathbb{BC} = \mathbb{C}(\mathbf{i})\mathbf{e}^+ + \mathbb{C}(\mathbf{i})\mathbf{e}^-. \tag{1.19}$$

Furthermore, we assume that the sequence c of coefficients has support on a set $T \subset \mathbb{Z}^2_\rho$ which is much smaller than the dimension $D = (2\rho + 1)^2$ of $\prod_\rho$. In other words, the finite combination in (1.18) is *sparse*, that is, only a few coefficients $c_{k,\tilde{k}}$ are non-zero.

As we do not possess any information on T except its maximum size, we introduce the auxiliary (non linear) space $\prod_\rho(M)$ of all polynomials of type (1.18) such that the sequence of coefficients c has support on a set $T \subset \mathbb{Z}_\rho^2$ satisfying to $|T| \leq M$, i.e., $f \in \prod_\rho(M) \subset \prod_\rho$ is of the form

$$f(x,y) = \sum_{(k,\tilde{k})\in T,\ |T|\leq M} c_{k,\tilde{k}} \left(e^{\mathbf{i}kx}\mathbf{e}^{+} + e^{\mathbf{i}\tilde{k}y}\mathbf{e}^{-}\right), \quad (x,y) \in [0,2\pi]^2, \tag{1.20}$$

and we say f has sparsity M.

We now state our problem: given a sampling set of N independent random variables having uniform distribution on $[0,2\pi]^2$, and assuming the signal f to be sparse, that is $f \in \prod_\rho(M)$, $(M \lll D)$, we wish to reconstruct f from its known samples at randomly chosen points with uniform distribution on the square $[0,2\pi]^2$.

1.2.2 Main theorems

As indicated before, using basis pursuit methods one can reconstruct the signal exactly. Unfortunately, this method can be too restrictive and computationally expensive for most applications. Therefore, we use an approach based on [69] to obtain a probabilistic answer to our problem. The theorems below are analogues to the ones in Candés, Romberg, and Tao [19] and in Rauhut [69]. First, we state the following ℓ_1-minimization problem: given a set of samples $\{f(x_j,y_j), j = 1,\dots,N\}$ we wish to find a sequence $c = (c_{k,\tilde{k}}) \in \ell_2(\mathbb{Z}_\rho^2)$ solving

$$\min \|c\|_{\ell_1} \tag{1.21}$$

$$s.t. \quad g(x_j,y_j) := \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} c_{k,\tilde{k}} \left(e^{\mathbf{i}kx_j}\mathbf{e}^{+} + e^{\mathbf{i}\tilde{k}y_j}\mathbf{e}^{-}\right) = f(x_j,y_j),\ j = 1,\dots,N,$$

where

$$\|c\|_{\ell_1} := \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} 2\mathrm{Re}(|c_{k,\tilde{k}}|_{\mathbf{k}}) = \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} (|c_k| + |c_{\tilde{k}}|).$$

The following theorems provide an answer as when the signal f can be reconstructed by means of solving this minimization problem.

Theorem 1.2.1. *Assume $f \in \prod_\rho(M)$ and let $X = \{(x_1,y_1),\dots,(x_N,y_N)\} \subset [0,2\pi]^2$ be a set of independent random variables having uniform distribution on $[0,2\pi]^2$.*

Choose $n \in \mathbb{N}$, $\beta > 0$, $\kappa > 0$ and $K_1,\cdots,K_n \in \mathbb{N}$ such that

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \ \ and \ \ \frac{\kappa}{1-\kappa} \leq \frac{1-a}{1+a} M^{-3/2}. \tag{1.22}$$

Set $\theta := N/M$. Then, f can be reconstructed exactly from its sample values $f(x_1,y_1),\dots,f(x_N,y_N)$

by solving the ℓ_1-minimization problem (1.21) with a probability of, at least,

$$1 - \left(D\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2}\, M\, G_{2n}(\theta) \right), \tag{1.23}$$

where, we recall, $D = (2\rho + 1)^2$.

While the above theorem provides exact constants we can give a version of the theorem which is somewhat easier to apply.

Theorem 1.2.2. *Assume $f \in \prod_\rho(M)$ and let $X = \{(x_1, y_1), \dots, (x_N, y_N)\} \subset [0, 2\pi]^2$ be a set of independent random variables having uniform distribution on $[0, 2\pi]^2$.*

If for some $\epsilon > 0$, there exists an absolute constant $C > 0$ such that it holds

$$N \geq CM \log(D/\epsilon), \tag{1.24}$$

then, with a probability at least $1 - \epsilon$, the signal f can be recovered from its sample values $f(x_1, y_1), \dots, f(x_N, y_N)$ by solving the minimization problem (1.21).

The proof of these two theorems requires several additional lemmas. These will be discussed in the next subsection. Although some parts follow closely Rauhut [69], we have to resort to our own tools for the bicomplex case, since the algebraic construction is a quite different one.

1.2.3 Additional Lemmas

As in [69], we begin by introducing some auxiliary notations: let $\ell_2(\mathbb{Z}_\rho^2)$, $\ell_2(T)$, and $\ell_2(X)$, denote the ℓ_2-spaces (w.r.t. the hyperbolic norm (1.9)) of sequences indexed by the grids $\mathbb{Z}_\rho^2$, T, and X, respectively.

We need the sampling operator $\mathcal{F}_X : \ell_2(\mathbb{Z}_\rho^2) \to \ell_2(X)$, given by

$$\mathcal{F}_X(c)(x_j, y_j) := \sum_{(k,\tilde{k}) \in \mathbb{Z}_\rho^2} c_{k,\tilde{k}} \left(e^{\mathbf{i}kx_j}\mathbf{e}^{+} + e^{\mathbf{i}\tilde{k}y_j}\mathbf{e}^{-} \right)_{j=1,\dots,N},$$

as well as its restriction, $\mathcal{F}_{TX}$, to sequences with support on T. Thus, $\mathcal{F}_{TX}$ is an operator acting from $\ell_2(T)$ in $\ell_2(X)$. Related to these operators, we have the adjoint operators, $\mathcal{F}_X^* : \ell_2(X) \to \ell_2(\mathbb{Z}_\rho^2)$ and $\mathcal{F}_{TX}^* : \ell_2(X) \to \ell_2(T)$.

As an abuse of language, and when clear from the context, we shall denote the sample sequence by the same letter of the original signal, that is to say, $f = (f(x_j, y_j))_{j=1,\dots,N} \in \ell_2(X)$.

As a starting point, we want to reconstruct a sequence $c \in \ell_2(\mathbb{Z}_\rho^2)$ from the data $\mathcal{F}_X c = f$, where $f = (f(x_j, y_j))_{j=1,\dots,N}$ denotes the sample sequence of the signal in $\ell_2(X)$, by solving the following ℓ_1-minimization problem

$$\min \|c\|_{\ell_1} \quad \text{subject to} \quad \mathcal{F}_X c = f. \tag{1.25}$$

Let us define the *sign function* in the bicomplex case as

$$\mathrm{sgn}\zeta = \mathrm{sgn}(\beta_1\mathbf{e}^+ + \beta_2\mathbf{e}^-) := (\mathrm{sgn}\beta_1)\mathbf{e}^+ + (\mathrm{sgn}\beta_2)\mathbf{e}^-,$$

where for each $\beta \in \mathbb{C}(\mathbf{i})$ we have $\mathrm{sgn}\beta = \frac{\beta}{|\beta|}$, if $\beta \neq 0$, and 0 otherwise.

For the sequence of coefficients $c := (c_{k,\tilde{k}})_{k,\tilde{k}} = (c_k\mathbf{e}^+ + c_{\tilde{k}}\mathbf{e}^-)_{k,\tilde{k}}$ it is obvious that if $(k,\tilde{k}) \notin \mathrm{supp}\, c$ then, $\mathrm{sgn}\, c_{k,\tilde{k}} = 0$ while $|\mathrm{sgn}\, c_{k,\tilde{k}}| = 1$ for all $(k,\tilde{k}) \in \mathrm{supp}\, c$.

Lemma 1.2.3. *Given two bicomplex numbers $f \neq 0$ and g it holds*

1. $|\mathrm{sgn} f|_{\mathbf{k}} = 1$*;*
2. $f\,\overline{\mathrm{sgn} f} = |f|_{\mathbf{k}}$*;*
3. $|g|_{\mathbf{k}}\, |\mathrm{sgn} f|_{\mathbf{k}} = |g\, \mathrm{sgn} f|_{\mathbf{k}}$.

Proof. Let us write $f = f_1\mathbf{e}^+ + f_2\mathbf{e}^-$ and $g = g_1\mathbf{e}^+ + g_2\mathbf{e}^-$. Due to (1.8) we have $|\mathrm{sgn} f|_{\mathbf{k}} = |\overline{\mathrm{sgn} f}|_{\mathbf{k}}$. Then,

1.

$$|\mathrm{sgn} f|_{\mathbf{k}} = \left|\frac{f_1\mathbf{e}^+ + f_2\mathbf{e}^-}{|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-}\right|_{\mathbf{k}} = \left|\left(f_1\mathbf{e}^+ + f_2\mathbf{e}^-\right)\left(|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-\right)^{-1}\right|_{\mathbf{k}}$$
$$= \left|\frac{f_1}{|f_1|}\mathbf{e}^+ + \frac{f_2}{|f_2|}\mathbf{e}^-\right|_{\mathbf{k}} = \frac{|f_1|}{|f_1|}\mathbf{e}^+ + \frac{|f_2|}{|f_2|}\mathbf{e}^- = 1.$$

2.

$$f\,\overline{\mathrm{sgn} f} = \left(f_1\mathbf{e}^+ + f_2\mathbf{e}^-\right)\frac{\overline{f_1}\mathbf{e}^+ + \overline{f_2}\mathbf{e}^-}{|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-} = \frac{|f_1|^2\mathbf{e}^+ + |f_2|^2\mathbf{e}^-}{|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-}$$
$$= |f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^- = |f|_{\mathbf{k}}.$$

Finally, for *3.*, we have for the left hand side,

$$|g|_{\mathbf{k}}\, |\mathrm{sgn} f|_{\mathbf{k}} = |g|_{\mathbf{k}} = |g_1|\mathbf{e}^+ + |g_2|\mathbf{e}^-,$$

while for the right hand side,

$$|g\,\mathrm{sgn} f|_{\mathbf{k}} = \left|\left(g_1\mathbf{e}^+ + g_2\mathbf{e}^-\right)\frac{f_1\mathbf{e}^+ + f_2\mathbf{e}^-}{|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-}\right|_{\mathbf{k}} = \left|\frac{f_1g_1\mathbf{e}^+ + f_2g_2\mathbf{e}^-}{|f_1|\mathbf{e}^+ + |f_2|\mathbf{e}^-}\right|_{\mathbf{k}}$$
$$= \left|\frac{f_1g_1}{|f_1|}\mathbf{e}^+ + \frac{f_2g_2}{|f_2|}\mathbf{e}^-\right|_{\mathbf{k}} = \frac{|f_1||g_1|}{|f_1|}\mathbf{e}^+ + \frac{|f_2||g_2|}{|f_2|}\mathbf{e}^-$$
$$= |g_1|\mathbf{e}^+ + |g_2|\mathbf{e}^- = |g|_{\mathbf{k}}.$$

□

The following lemma is an adaptation to our setting of [19], Lemma 2.1.

Lemma 1.2.4. *Let $c = (c_{k,\tilde{k}}) \in \ell_2(\mathbb{Z}_\rho^2)$ be a sequence with support on T and satisfying to $\mathcal{F}_{TX}c = f$. Also, assume $\mathcal{F}_{TX} : \ell_2(T) \to \ell_2(X)$ to be injective.*

If there exists a sequence $P \in \ell_2(\mathbb{Z}_\rho^2)$ with the following properties:
(i) $P_{k\tilde{k}} = \operatorname{sgn}\left(c_{k\tilde{k}}\right)_{k\tilde{k}}$ for all $(k,\tilde{k}) \in T$;
(ii) $\left|P_{k\tilde{k}}\right| < 1$ for all $(k,\tilde{k}) \notin T$;
(iii) there exists a $\lambda = \lambda_1\mathbf{e}^+ + \lambda_2\mathbf{e}^- \in \ell_2(X)$ such that $P = [\mathcal{F}_X^*]_1\,\lambda_1\mathbf{e}^+ + [\mathcal{F}_X^*]_2\,\lambda_2\mathbf{e}^-$,
then c is the unique minimizer to problem (1.25).

Proof. First, we eliminate the trivial cases where X and T are empty. Also, we assume that $f = (f(x_j, y_j), j = 1, \ldots, N)$ is a non-zero vector.

Let $h = (h_{k,\tilde{k}}) \in \ell_2(\mathbb{Z}_\rho^2)$ be an arbitrary vector such that $\mathcal{F}_X h = \mathcal{F}_X c$. Take $g = h - c$. Then, $\mathcal{F}_X g = 0$ on X. Taking in account Lemma 1.2.3, we get for all $(k,\tilde{k}) \in T$ that

$$\begin{aligned} |h_{k,\tilde{k}}|_{\mathbf{k}} &= |c_{k,\tilde{k}} + g_{k,\tilde{k}}|_{\mathbf{k}} \\ &= |(c_{k,\tilde{k}} + g_{k,\tilde{k}})\overline{\operatorname{sgn} c_{k,\tilde{k}}}\,\operatorname{sgn} c_{k,\tilde{k}}|_{\mathbf{k}} \quad \text{by } \mathit{1.} \\ &= |\left(c_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}} + g_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}\right)\operatorname{sgn} c_{k,\tilde{k}}|_{\mathbf{k}} \\ &= |\,|c_{k,\tilde{k}}|_{\mathbf{k}} + g_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}|_{\mathbf{k}}|\overline{\operatorname{sgn} c_{k,\tilde{k}}}|_{\mathbf{k}} \quad \text{by } \mathit{2.} \text{ and } \mathit{3.} \\ &= |\,|c_{k,\tilde{k}}|_{\mathbf{k}} + g_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}|_{\mathbf{k}}, \quad \text{by } \mathit{1.} \text{ again.} \end{aligned} \tag{1.26}$$

We apply the idempotent decomposition (1.6) to each sequence, namely,

$$c_{k,\tilde{k}} = c_k\mathbf{e}^+ + c_{\tilde{k}}\mathbf{e}^-, \quad g_{k,\tilde{k}} = g_k\mathbf{e}^+ + g_{\tilde{k}}\mathbf{e}^-, \quad h_{k,\tilde{k}} = h_k\mathbf{e}^+ + h_{\tilde{k}}\mathbf{e}^-.$$

Recall the hyperbolic norm (1.9)

$$2\mathrm{Re}\left(|z_1\mathbf{e}^+ + z_2|\mathbf{e}^-|_{\mathbf{k}}\right) = 2\mathrm{Re}\left(|z_1|\mathbf{e}^+ + |z_2|\mathbf{e}^-\right) = |z_1| + |z_2|,$$

for all $z_1, z_2 \in \mathbb{C}(\mathbf{i})$. Based on relation (1.8) expression (1.26) can be further simplified as

$$\begin{aligned} 2\mathrm{Re}\left(|h_{k,\tilde{k}}|_{\mathbf{k}}\right) &= 2\mathrm{Re}\left(|\,|c_{k,\tilde{k}}|_{\mathbf{k}} + g_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}|_{\mathbf{k}}\right) \\ &= 2\mathrm{Re}\left(|\,|c_k|\mathbf{e}^+ + |c_{\tilde{k}}|\mathbf{e}^- + g_k\,\overline{\operatorname{sgn} c_k}\mathbf{e}^+ + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}\mathbf{e}^-|_{\mathbf{k}}\right) \\ &= 2\mathrm{Re}\left(|\left(|c_k| + g_k\,\overline{\operatorname{sgn} c_k}\right)\mathbf{e}^+ + \left(|c_{\tilde{k}}| + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}\right)\mathbf{e}^-|_{\mathbf{k}}\right) \\ &= |\,|c_k| + g_k\,\overline{\operatorname{sgn} c_k}| + |\,|c_{\tilde{k}}| + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}|. \end{aligned}$$

Now,

$$\begin{aligned} |\,|c_k| + g_k\,\overline{\operatorname{sgn} c_k}| + |\,|c_{\tilde{k}}| + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}| &\geq |c_k| + \mathrm{Re}\left(g_k\,\overline{\operatorname{sgn} c_k}\right) + |c_{\tilde{k}}| + \mathrm{Re}\left(g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}\right) \\ &= |c_k| + |c_{\tilde{k}}| + \mathrm{Re}\left(g_k\,\overline{\operatorname{sgn} c_k} + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}\right) \\ &= |c_k| + |c_{\tilde{k}}| + 2\mathrm{Re}\left(g_k\,\overline{\operatorname{sgn} c_k}\mathbf{e}^+ + g_{\tilde{k}}\,\overline{\operatorname{sgn} c_{\tilde{k}}}\mathbf{e}^-\right) \\ &= |c_k| + |c_{\tilde{k}}| + 2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}\right). \end{aligned}$$

If a sequence P exists satisfying *(i)-(iii)* then

$$2\mathrm{Re}(|h_{k,\tilde{k}}|_{\mathbf{k}}) \geq |c_k| + |c_{\tilde{k}}| + 2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{\mathrm{sgn} c_{k,\tilde{k}}}\right) = |c_k| + |c_{\tilde{k}}| + 2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right).$$

Thus, we have obtained that for all $(k, \tilde{k}) \in T$ we have

$$|c_k| + |c_{\tilde{k}}| + 2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right) \leq |h_k| + |h_{\tilde{k}}| = 2\mathrm{Re}\left(|h_{k,\tilde{k}}|_{\mathbf{k}}\right).$$

Moreover, for $(k, \tilde{k}) \notin T$ we have by direct calculations,

$$2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right) \leq 2\mathrm{Re}\left(|g_{k,\tilde{k}}|_{\mathbf{k}}\right) = 2\mathrm{Re}\left(|h_{k,\tilde{k}}|_{\mathbf{k}}\right)$$

due to *(ii)*. Hence, we obtain

$$\|h\|_{\ell_1} \geq \|c\|_{\ell_1} + \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} 2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right).$$

By the Plancherel formula for the discrete Fourier transform (see [40], page 3), we have

$$\sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} \mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right) = \frac{1}{N}\sum_{j=1}^{N} \mathrm{Re}\left(\mathcal{F}_X g(x_j, y_j)\,\overline{\mathcal{F}_X P(x_j, y_j)}\right). \tag{1.27}$$

Let us take a look closer into this expression. Taking in account that

$$g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}} = g_k\,\overline{P_k}\mathbf{e}^+ + g_{\tilde{k}}\,\overline{P_{\tilde{k}}}\mathbf{e}^-,$$

we obtain

$$\begin{aligned}
\sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} \mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right) &= \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} \mathrm{Re}\left(g_k\,\overline{P_k}\mathbf{e}^+ + g_{\tilde{k}}\,\overline{P_{\tilde{k}}}\mathbf{e}^-\right) \\
&= \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} \mathrm{Re}\left(g_k\,\overline{[\mathcal{F}_X^*]_1\lambda_1]_k}\mathbf{e}^+ + g_{\tilde{k}}\,\overline{[\mathcal{F}_X^*]_2\lambda_2]_{\tilde{k}}}\mathbf{e}^-\right) \\
&= \frac{1}{N}\sum_{j=1}^{N} \mathrm{Re}\left(\mathcal{F}_X g(x_j, y_j)\,\overline{\lambda(x_j, y_j)}\right) = 0,
\end{aligned}$$

since $\mathcal{F}_X g(x_j, y_j) = 0$, $j = 1, \ldots, N$. Therefore, $\|h\|_{\ell_1} \geq \|c\|_{\ell_1}$ and equality holds for $2\mathrm{Re}\left(g_{k,\tilde{k}}\,\overline{P_{k,\tilde{k}}}\right) = 2\mathrm{Re}\left(|g_{k,\tilde{k}}|_{\mathbf{k}}\right)$. Since $|P_{k,\tilde{k}}| < 1$, this forces g to vanish outside T. Taking into account the injectivity of $\mathcal{F}_{TX}$ we get that if $\mathcal{F}_X g$ vanishes on X, g vanishes identically and, therefore, $h = c$. Thus, this shows that c is the unique minimizer to the problem (1.25). □

In order to later on prove Theorem 1.2.1 we need to show that with high probability there exists a sequence P satisfying to *(i)-(iii)* of Lemma 1.2.4.

As in [69], we introduce the restriction operator $R_T : \ell_2(\mathbb{Z}_\rho^2) \to \ell_2(T)$, which restricts the sequence $c \in \ell_2(\mathbb{Z}_\rho^2)$ to $\ell_2(T)$ by $R_T c := (c_{k,\tilde{k}})_{(k,\tilde{k})\in T}$. Its adjoint, or extension operator, $R_T^* = E_T : \ell_2(T) \to \ell_2(\mathbb{Z}_\rho^2)$ is the operator which extends the restricted sequence $(c_{k,\tilde{k}})_{(k,\tilde{k})\in T}$ to a sequence $d \in \ell_2(\mathbb{Z}_\rho^2)$ where $d_{k,\tilde{k}} = c_{k,\tilde{k}}$ for $(k,\tilde{k}) \in T$, and $d_{k,\tilde{k}} = 0$ otherwise.

Now, we assume that $\mathcal{F}_{TX}^*\mathcal{F}_{TX} : \ell_2(T) \to \ell_2(T)$ is invertible. This is true almost surely if $N \geq |T|$ since then $\mathcal{F}_{TX}$ is injective. The following lemma, adapted from [53], give us the inverse of this matrix.

Lemma 1.2.5. *For a non-singular matrix $\mathcal{F}_{TX}^*\mathcal{F}_{TX}$ we have*

$$(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1} = [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_1^{-1}\,\mathbf{e}^+ + [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_2^{-1}\,\mathbf{e}^-.$$

Since the proof is straight forward we will omit it here. We proceed with an explicit construction of P by

$$P := \mathcal{F}_X^*\mathcal{F}_{TX}(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}c,$$

where as before $T = \mathrm{supp}(c)$ and $\mathrm{sgn}c_{k\tilde{k}} = \mathrm{sgn}\,c_k\mathbf{e}^+ + \mathrm{sgn}\,c_{\tilde{k}}\mathbf{e}^-$. Clearly, P verifies *(i)* and *(iii)* of Lemma 1.2.4, with

$$\lambda := \mathcal{F}_{TX}(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}c \in \ell_2(X).$$

It remains to prove that P satisfies *(ii)* with high probability. We begin with the idempotent decomposition of our bicomplex matrices

$$I := I\mathbf{e}^+ + I\mathbf{e}^- \text{ (identity matrix in } \ell_2(T)), \quad E_T := E_T\mathbf{e}^+ + E_T\mathbf{e}^-,$$

$$\mathcal{F}_X^*\mathcal{F}_{TX} := [\mathcal{F}_X^*\mathcal{F}_{TX}]_1\,\mathbf{e}^+ + [\mathcal{F}_X^*\mathcal{F}_{TX}]_2\,\mathbf{e}^-,$$

and

$$\mathcal{F}_{TX}^*\mathcal{F}_{TX} := [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_1\,\mathbf{e}^+ + [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_2\,\mathbf{e}^-.$$

Next, we introduce the auxiliary operators

$$H := \ell_2(T) \to \ell_2(\mathbb{Z}_\rho^2), \quad H := NE_T - \mathcal{F}_X^*\mathcal{F}_{TX}, \tag{1.28}$$

and

$$H_0 := R_T H = \ell_2(T) \to \ell_2(T), \quad H_0 := NI - \mathcal{F}_{TX}^*\mathcal{F}_{TX}. \tag{1.29}$$

Obviously, H_0 is self-adjoint, and both operators admit the following idempotent decompositions

$$\begin{aligned} H &:= N\left[E_T\mathbf{e}^+ + E_T\mathbf{e}^-\right] - \left[[\mathcal{F}_X^*\mathcal{F}_{TX}]_1\,\mathbf{e}^+ + [\mathcal{F}_X^*\mathcal{F}_{TX}]_2\,\mathbf{e}^-\right] \\ &= (NE_T - [\mathcal{F}_X^*\mathcal{F}_{TX}]_1)\,\mathbf{e}^+ + (NE_T - [\mathcal{F}_X^*\mathcal{F}_{TX}]_2)\,\mathbf{e}^- \end{aligned}$$

and

$$\begin{aligned} H_0 &:= N\left[I\mathbf{e}^+ + I\mathbf{e}^-\right] - \left[[\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_1\,\mathbf{e}^+ + [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_2\,\mathbf{e}^-\right] \\ &= (NI - [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_1)\,\mathbf{e}^+ + (NI - [\mathcal{F}_{TX}^*\mathcal{F}_{TX}]_2)\,\mathbf{e}^-. \end{aligned}$$

We are particularly interested in the entries of these matrices at a position $(\ell,\tilde{\ell}),(k,\tilde{k})$. We get

$$H_0\left[(\ell,\tilde{\ell}),(k,\tilde{k})\right] = H_{0,1}(\ell,k)\mathbf{e}^+ + H_{0,2}(\tilde{\ell},\tilde{k})\mathbf{e}^-,$$

and

$$H\left[[(\ell,\tilde{\ell}),(k,\tilde{k})\right] = H_1(\ell,k)\mathbf{e}^+ + H_2(\tilde{\ell},\tilde{k})\mathbf{e}^-,$$

where

$$H_1(\ell,k) := (1-\delta_{\ell,k})\sum_{j=1}^{N} e^{\mathbf{i}(\ell-k)x_j}, \quad H_2(\tilde{\ell},\tilde{k}) := (1-\delta_{\tilde{\ell},\tilde{k}})\sum_{j=1}^{N} e^{\mathbf{i}(\tilde{\ell}-\tilde{k})y_j}.$$

Thus, we have the matrix

$$H = \left[(1-\delta_{\ell,k})\sum_{j=1}^{N} e^{\mathbf{i}(\ell-k)x_j}\mathbf{e}^+ + (1-\delta_{\tilde{\ell},\tilde{k}})\sum_{j=1}^{N} e^{\mathbf{i}(\tilde{\ell}-\tilde{k})y_j}\mathbf{e}^-\right]_{(\ell,\tilde{\ell}),(k,\tilde{k})\in\mathbb{Z}_\rho^2}$$

acting on a sequence $c = (c_{k,\tilde{k}})$ (we repeat, with support in T) as

$$(Hc)_{\ell\tilde{\ell}} = -\sum_{j=1}^{N}\sum_{(k,\tilde{k})\in T}\left(c_k e^{\mathbf{i}(k-\ell)x_j}\mathbf{e}^+ + c_{\tilde{k}} e^{\mathbf{i}(\tilde{k}-\tilde{\ell})y_j}\mathbf{e}^-\right). \tag{1.30}$$

Using these operators and Lemma 1.2.5, we get

$$\begin{aligned} P &= \mathcal{F}_X^*\mathcal{F}_{TX}(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}c \\ &= (NE_T - H)(NI - H_0)^{-1}R_T\mathrm{sgn}c \end{aligned} \tag{1.31}$$

Since we aim to prove that P satisfy *(ii)* of Lemma 1.2.4 we restrict ourselves to $T^c := \mathbb{Z}_\rho^2 \setminus T$. In this case, $R_{T^c}E_T = 0$ and

$$(R_{T^c}P)_{k,\tilde{k}} = P_{k,\tilde{k}} = -\left(H\,(NI - H_0)^{-1}\,R_T\mathrm{sgn}c\right)_{k,\tilde{k}}, \tag{1.32}$$

for all $(k,\tilde{k}) \in T^c$. Let us take a closer look at the quantity $\left(I - \frac{1}{N}H_0\right)^{-1}$. Due to $(1-\gamma)^{-1} = (1-\gamma^n)^{-1}(1+\gamma+\cdots+\gamma^{n-1})$ we can write

$$\left(I - \frac{1}{N}H_0\right)^{-1} = \left(I - \left(\frac{1}{N}H_0\right)^n\right)^{-1}\sum_{m=0}^{n-1}\left(\frac{1}{N}H_0\right)^m,$$

while by the von Neumann series we have

$$\left(I-\left(\frac{1}{N}H_0\right)^n\right)^{-1}=I+\sum_{r=1}^{\infty}\left(\frac{1}{N}H_0\right)^{rn}:=I+A_n \tag{1.33}$$

under the assumption that the last series converges. Therefore, for all $(k,\tilde{k})\in T^c$ we get

$$P_{k,\tilde{k}}=-\left(\frac{1}{N}H(I+A_n)\sum_{m=0}^{n-1}\left(N^{-1}H_0\right)^m R_T\mathrm{sgnc}\right)_{k,\tilde{k}}=-(P^{(1)}+P^{(2)})_{k,\tilde{k}},$$

where (recall, $H_0=R_TH$)

$$P^{(1)}=\sum_{m=1}^{n}\left(N^{-1}HR_T\right)^m\mathrm{sgnc},\quad P^{(2)}=\frac{1}{N}HA_nR_T\left(I+\sum_{m=1}^{n-1}\left(N^{-1}HR_T\right)^m\right)\mathrm{sgnc}.$$

Our goal is show that the probability of $|P_{k,\tilde{k}}|<1$ is low for $(k,\tilde{k})$ outside T. For that effect, we estimate

$$\mathbb{P}\left(\sup_{(k,\tilde{k})\in T^c}|P_{k,\tilde{k}}|\geq 1\right).$$

Let $a_1,a_2>0$ be non-negative real numbers satisfying to $a_1+a_2=1$. Then,

$$\mathbb{P}\left(\sup_{(k,\tilde{k})\in T^c}|P_{k,\tilde{k}}|\geq 1\right)\leq\mathbb{P}\left(\left\{\sup_{(k,\tilde{k})\in T^c}|P^{(1)}_{k,\tilde{k}}|\geq a_1\right\}\cup\left\{\sup_{(k,\tilde{k})\in T^c}|P^{(2)}_{k,\tilde{k}}|\geq a_2\right\}\right). \tag{1.34}$$

Now, we have

$$\begin{aligned}\mathbb{P}\left(|P^{(1)}_{k,\tilde{k}}|\geq a_1\right) &= \mathbb{P}\left(\left|\left(\sum_{m=1}^{n}\left(N^{-1}HR_T\right)^m\mathrm{sgnc}\right)_{k,\tilde{k}}\right|\geq a_1\right)\\ &\leq \mathbb{P}\left(\sum_{m=1}^{n}\left|\left(\left(N^{-1}HR_T\right)^m\mathrm{sgnc}\right)_{k,\tilde{k}}\right|\geq a_1\right):=\mathbb{P}(E_{k,\tilde{k}}).\end{aligned} \tag{1.35}$$

For the term $P^{(2)}$ we obtain

$$\sup_{(k,\tilde{k})\in T^c}\left|P^{(2)}_{k,\tilde{k}}\right|\leq\|P^{(2)}\|_\infty\leq\|N^{-1}HA_n\|_\infty\left(1+\left\|R_T\sum_{m=1}^{n-1}\left(N^{-1}HR_T\right)\mathrm{sgnc}\right\|_{\ell^\infty(T)}\right). \tag{1.36}$$

where $\|\cdot\|_\infty:=\|\cdot\|_{\ell^\infty(T)\to\ell^\infty(\mathbb{Z}^2_\rho)}$, and $\ell^\infty(\mathbb{Z}^2_\rho)$ denote the space of sequences indexed by $\mathbb{Z}^2_\rho$ with the supremum hyperbolic norm. We have, by (1.35),

$$\mathbb{P}\left(\left|\left(\sum_{m=1}^{n-1}\left(N^{-1}HR_T\right)^m\mathrm{sgnc}\right)_{k,\tilde{k}}\right|\geq a_1\right)\leq\mathbb{P}\left(\sum_{m=1}^{n-1}\left|\left(\left(N^{-1}HR_T\right)^m\mathrm{sgnc}\right)_{k,\tilde{k}}\right|\geq a_1\right)$$

$$\leq \mathbb{P}\left(\sum_{m=1}^{n}\left|\left(\left(N^{-1}HR_T\right)^m \mathrm{sgnc}\right)_{k,\tilde{k}}\right| \geq a_1\right) = \mathbb{P}(E_{k,\tilde{k}}).$$

Now, we analyse the operator norm in (1.36) in terms of the Frobenius norm (see [59] and [45]). First of all, we have

$$\left\|N^{-1}HA_n\right\|_\infty \leq \left\|N^{-1}H\right\|_\infty \|A_n\|_{\ell^\infty(T)} = \left\|N^{-1}H\right\|_\infty \left\|\sum_{r=1}^{\infty}\left(N^{-1}H_0\right)^{rn}\right\|_{\ell^\infty(T)}. \tag{1.37}$$

Given a square matrix B, with bicomplex entries, it holds for its Frobenius norm defined with respect to the hyperbolic norm, the equality

$$\|B\|_F^2 := \mathrm{Re}\left(\mathrm{tr}(BB^*)\right) = \sum_{r,s} 2\mathrm{Re}\left(|B_{rs}|_{\mathbf{k}}^2\right),$$

where $\mathrm{tr}(BB^*)$ denotes the trace of BB^*. Assume now that

$$\left\|\left(\frac{1}{N}H_0\right)^n\right\|_F \leq \kappa < 1. \tag{1.38}$$

From (1.33) we obtain

$$\|A_n\|_F = \left\|\sum_{r=1}^{\infty}\left(\frac{1}{N}H_0\right)^{rn}\right\|_F \leq \sum_{r=1}^{\infty}\left\|(N^{-1}H_0)^n\right\|_F^r \leq \sum_{r=1}^{\infty}\kappa^r = \frac{\kappa}{1-\kappa}.$$

Also, by Cauchy-Schwarz inequality we get

$$\|A_n\|_{\ell^\infty(T)}^2 \leq |T|\left(2\sup_r \sum_s \mathrm{Re}\left(|[A_n]_{r,s}|_{\mathbf{k}}\right)^2\right) \leq |T|\,\|A_n\|_F^2. \tag{1.39}$$

Under the assumptions (1.38), and $\left\|\sum_{m=1}^{n-1}\left(\frac{1}{N}HR_T\right)^m \mathrm{sgnc}\right\|_\infty < a_1$, it holds

$$\sup_{(k,\tilde{k})\in T^c} |P_{k\tilde{k}}^{(2)}| \leq (1+a_1)\frac{\kappa}{1-\kappa}\,|T|^{\frac{3}{2}}. \tag{1.40}$$

Moreover, if

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}\,|T|^{-\frac{3}{2}} \tag{1.41}$$

then, we have $\sup_{(k,\tilde{k})\in T^c} |P_{k\tilde{k}}^{(2)}| \leq a_2$.

Also, it follows from (1.41) that $\kappa < 1$ and $|T| \geq 1$ (note that if $T = \emptyset$ then $c = 0$ and obviously the ℓ_1-minimization problem recovers the sequence). This concludes the proof of the existence with high probability of a sequence P satisfying to the conditions of Lemma 1.2.4. As an additional consequence, it also ensures that $\mathcal{F}_{TX}$ is injective almost surely.

Next, we proceed with the analysis of the above probability bearing in mind that T is a deterministic variable, as seen in Theorem 1.2.2. Using (1.34) we have

$$\mathbb{P}\left(\sup_{(k,\tilde{k})\in T^c}|P_{k,\tilde{k}}|\geq 1\right) \leq \sum_{(k,\tilde{k})\in T^c}\mathbb{P}\left(E_{k,\tilde{k}}\right)+\mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\|_F\geq\kappa\right). \tag{1.42}$$

In conclusion, the probability of the existence of a sequence P depends on the estimates for $\mathbb{P}(E_{k\tilde{k}})$ and for $\mathbb{P}\big(\left\|\left(N^{-1}H_0\right)^n\right\|_F\geq\kappa\big)$.

1.2.4 Analysis of the Powers H_0^{2n}

We now compute the expectation of $\|H_0^n\|_F^2$ with respect to the sampling set $\{(x_j,y_j), j=1,\dots,N\}$.

Lemma 1.2.6. *It holds*

$$\mathbb{E}_X\left[\|H_0^n\|_F^2\right]=\sum_{t=1}^{\min\{n,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2n,t)}\sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T\\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r\\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}}$$
$$\times\prod_{A\subset\mathcal{A}}\left(\delta\left(\sum_{r\in A}(k_{r+1}-k_r)\right)+\delta\left(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)\right)\right)$$

where $P(2n,t)$ denotes the set of all partitions $A=(A_1,...,A_t)$ of the set $1,...,2n$.

Proof. We remember the operator (1.29) with an idempotent decomposition at each entry $[(\ell,\tilde{\ell}),(k,\tilde{k})]$

$$H_0\left[(\ell,\tilde{\ell}),(k,\tilde{k})\right]=H_{0,1}(\ell,k)\mathbf{e}^+ + H_{0,2}(\tilde{\ell},\tilde{k})\mathbf{e}^-.$$

Hence, its adjoint is

$$H_0^*\left[(\ell,\tilde{\ell}),(k,\tilde{k})\right]=\overline{H}_{0,1}(k,\ell)\mathbf{e}^+ + \overline{H}_{0,2}(\tilde{k},\tilde{\ell})\mathbf{e}^-,$$

where $\overline{H}_{0,1},\overline{H}_{0,2}$ represent the matrices with complex conjugated entries. Therefore, at the main diagonal we have

$$H_0^2\left[(k,\tilde{k}),(k,\tilde{k})\right]=\sum_{(\ell,\tilde{\ell})\in T}H_0\left[(k,\tilde{k}),(\ell,\tilde{\ell})\right]H_0^*\left[(\ell,\tilde{\ell}),(k,\tilde{k})\right]$$
$$=\sum_{(\ell,\tilde{\ell})\in T}\left(H_{0,1}(k,\ell)\mathbf{e}^+ + H_{0,2}(\tilde{k},\tilde{\ell})\mathbf{e}^-\right)\left(\overline{H}_{0,1}(k,\ell)\mathbf{e}^+ + \overline{H}_{0,2}(\tilde{k},\tilde{\ell})\mathbf{e}^-\right)$$
$$=\sum_{j_1,j_2=1}^{N}\sum_{\substack{(\ell,\tilde{\ell})\in T\\ \ell\neq k,\tilde{\ell}\neq\tilde{k}}}\left[e^{\mathbf{i}(\ell-k)x_{j_1}}e^{\mathbf{i}(k-\ell)x_{j_2}}\mathbf{e}^+ + e^{\mathbf{i}(\tilde{\ell}-\tilde{k})y_{j_1}}e^{\mathbf{i}(\tilde{k}-\tilde{\ell})y_{j_2}}\mathbf{e}^-\right]. \tag{1.43}$$

For the generalization to an arbitrary power $H_0^{2n} = (H_0 H_0^*)^n$, we shall calculate its trace as the sum of the entries $\left[(k_1,\tilde{k}_1),(k_1,\tilde{k}_1)\right]$, and will denote the auxiliary entries $(\ell,\tilde{\ell})$ by $(k_s,\tilde{k}_s)$, $s = 2,\ldots,2n$. Hence, (1.43) becomes

$$H_0^2\left[(k_1,\tilde{k}_1),(k_1,\tilde{k}_1)\right]$$

$$= \sum_{j_1,j_2=1}^{N} \sum_{\substack{(k_2,\tilde{k}_2)\in T \\ k_2\neq k_1,\tilde{k}_2\neq\tilde{k}_1}} \left[e^{\mathbf{i}(k_2-k_1)x_{j_1}} e^{\mathbf{i}(k_1-k_2)x_{j_2}}\mathbf{e}^+ + e^{\mathbf{i}(\tilde{k}_2-\tilde{k}_1)y_{j_1}} e^{\mathbf{i}(\tilde{k}_1-\tilde{k}_2)y_{j_2}}\mathbf{e}^-\right],$$

while for the n-th power of our matrix we get

$$H_0^{2n}\left[(k_1,\tilde{k}_1),(k_1,\tilde{k}_1)\right] = \left(\sum_{(\ell,\tilde{\ell})\in T} H_0\left[(k_1,\tilde{k}_1),(\ell,\tilde{\ell})\right] H_0^*\left[(\ell,\tilde{\ell}),(k_1,\tilde{k}_1)\right]\right)^n$$

$$= \sum_{\substack{(k_2,\tilde{k}_2),\cdots,(k_{2n},\tilde{k}_{2n})\in T \\ k_2\neq k_1,\tilde{k}_2\neq\tilde{k}_1 \\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r,\ r=2,\cdots,2n-1 \\ k_{2n}\neq k_1,\tilde{k}_{2n}\neq\tilde{k}_1}} H_0^2\left[(k_1,\tilde{k}_1),(k_2,\tilde{k}_2)\right]\cdots H_0^2\left[(k_{2n},\tilde{k}_{2n}),(k_1,\tilde{k}_1)\right],$$

and the trace of $H_0^{2n} = (H_0 H_0^*)^n$ is then,

$$\mathrm{tr} H_0^{2n} = \sum_{(k_1,\tilde{k}_1)\in T} H_0^{2n}\left[(k_1,\tilde{k}_1),(k_1,\tilde{k}_1)\right]$$

$$= \sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T \\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r,\ r=1,\cdots,2n \\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}} H_0^2\left[(k_1,\tilde{k}_1),(k_2,\tilde{k}_2)\right]\cdots H_0^2\left[(k_{2n},\tilde{k}_{2n}),(k_1,\tilde{k}_1)\right]$$

$$= \sum_{j_1,\cdots,j_{2n}=1}^{N} \sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T \\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r \\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}}$$

$$\left[e^{\mathbf{i}(k_{r+1}-k_r)x_{j_r}} e^{\mathbf{i}(k_r-k_{r+1})x_{j_{r+1}}}\mathbf{e}^+ + e^{\mathbf{i}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_r}} e^{\mathbf{i}(\tilde{k}_r-\tilde{k}_{r+1})y_{j_{r+1}}}\mathbf{e}^-\right]$$

$$= \sum_{j_1,\cdots,j_{2n}=1}^{N} \sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T \\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r \\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}} \left[e^{\mathbf{i}\sum_{r=1}^{2n}(k_{r+1}-k_r)x_{j_r}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{r=1}^{2n}\tilde{k}_{r+1}-\tilde{k}_r)y_{j_r}}\mathbf{e}^-\right].$$

In consequence, its mean value

$$\mathbb{E}_X\left[\|H_0^n\|_F^2\right] = \mathbb{E}_X\left[2\mathrm{Re}\left(\mathrm{tr}((H_0 H_0^*)^n)\right)\right]$$

$$= \sum_{\substack{j_1=1\\ \vdots \\ j_{2n}=1}}^{N} \sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T\\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r\\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}} \mathbb{E}_X\left[2\mathrm{Re}\left(e^{\mathbf{i}\sum_{r=1}^{2n}(k_{r+1}-k_r)x_{j_r}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{r=1}^{2n}\tilde{k}_{r+1}-\tilde{k}_r)y_{j_r}}\mathbf{e}^-\right)\right].$$

Now, we have to keep in mind that some of the indices j_r might coincide, in which case the multiplying terms $(k_{r+1}-k_r)$, etc. should be added. The idea to solve this problem comes from [69] where the author introduced a rearrangement based on set partitions. Here, we follow that idea by associating a partition $\mathcal{A}=(A_1,\ldots,A_t)$ of the set $\{1,\ldots,2n\}$ to a certain vector $(j_1,\ldots,j_{2n})$ such that $j_r=j_{r'}$ if and only if r and r' are contained in the same set $A\subset\mathcal{A}$. This will allow us, in an unambiguous way, to write j_A instead of j_r if $r\in A$. The independence of the sampling variables leads to

$$\begin{aligned}
&\mathbb{E}_X\left[2\mathrm{Re}\left(e^{\mathbf{i}\sum_{r=1}^{2n}(k_{r+1}-k_r)x_{j_r}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{r=1}^{2n}\tilde{k}_{r+1}-\tilde{k}_r)y_{j_r}}\mathbf{e}^-\right)\right]\\
&=\mathbb{E}_X\left[2\mathrm{Re}\left(e^{\mathbf{i}\sum_{A\subset\mathcal{A}}\sum_{r\in A}(k_{r+1}-k_r)x_{j_A}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{A\subset\mathcal{A}}\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_A}}\mathbf{e}^-\right)\right]\\
&=\mathbb{E}_X\left[2\mathrm{Re}\left(\prod_{A\subset\mathcal{A}} e^{\mathbf{i}\sum_{r\in A}(k_{r+1}-k_r)x_{j_A}}\mathbf{e}^+ + \prod_{A\subset\mathcal{A}} e^{\mathbf{i}(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_A}}\mathbf{e}^-\right)\right]\\
&=\prod_{A\subset\mathcal{A}}\mathbb{E}_X\left[2\mathrm{Re}\left(e^{\mathbf{i}\sum_{r\in A}(k_{r+1}-k_r)x_{j_A}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_A}}\mathbf{e}^-\right)\right]
\end{aligned} \tag{1.44}$$

Finally, taking into account that the variables (x_{j_A}, x_{j_A}) have uniform distribution on $[0,2\pi]^2$ we conclude that the expectation value is

$$\begin{aligned}
&\mathbb{E}_X\left[2\mathrm{Re}\left(e^{\mathbf{i}\sum_{r\in A}(k_{r+1}-k_r)x_{j_A}}\mathbf{e}^+ + e^{\mathbf{i}(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_A}}\mathbf{e}^-\right)\right]\\
&=\mathbb{E}_X\left[e^{\mathbf{i}\sum_{r\in A}(k_{r+1}-k_r)x_{j_A}}\right]+\mathbb{E}_X\left[e^{\mathbf{i}(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y_{j_A}}\right]\\
&=\frac{1}{2\pi}\int_0^{2\pi} e^{\mathbf{i}\sum_{r\in A}(k_{r+1}-k_r)x}dx+\frac{1}{2\pi}\int_0^{2\pi} e^{\mathbf{i}(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)y}dy\\
&=\delta\left(\sum_{r\in A}(k_{r+1}-k_r)\right)+\delta\left(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)\right).
\end{aligned} \tag{1.45}$$

If $A\subset\mathcal{A}$ contains only one element then (1.45) vanishes taking in account our conditions $k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r$. Hence, $|A|>1$ for all $A\in\mathcal{A}$, i.e., partitions in $P(2n,t)$. Furthermore, note that for each t the number of vectors $(j_{A_1},\ldots,j_{A_t})\in\{1,\ldots,N\}^t$ with different entries is precisely

$$N(N-1)\cdots(N-t+1)=\frac{N!}{(N-t)!}$$

if $N\geq t$ and 0 if $N<t$. $\square$

For later reference we define the following quantity:

$$C(\mathcal{A},T) := \sum_{\substack{(k_1,\tilde{k}_1),\cdots,(k_{2n},\tilde{k}_{2n})\in T\\ k_{r+1}\neq k_r,\tilde{k}_{r+1}\neq\tilde{k}_r\\ k_{2n+1}:=k_1,\tilde{k}_{2n+1}:=\tilde{k}_1}} \prod_{A\subset\mathcal{A}} \left(\delta\left(\sum_{r\in A}(k_{r+1}-k_r)\right)+\delta\left(\sum_{r\in A}(\tilde{k}_{r+1}-\tilde{k}_r)\right)\right) \tag{1.46}$$

1.2.5 Further Necessary Estimates

In order to obtain estimates for $\mathbb{P}(E_{k,\tilde{k}})$ in (1.42) we require a priori estimations for the expectation of $\mathrm{Re}\left(\left|((HR_T)^m\mathrm{sgn}(c))_{k,\tilde{k}}\right|_{\mathbf{k}}^{2K}\right)$.

Lemma 1.2.7. *Let $c := c_1\mathbf{e}^+ + c_2\mathbf{e}^- \in \ell_2(\mathbb{Z}_\rho^2)$ be a sequence with support $\operatorname{supp} c = T$. Then, for every $(k,\tilde{k})\in\mathbb{Z}_\rho^2$ it holds*

$$\begin{aligned}
&\mathbb{E}_X\left[2\mathrm{Re}\left(\left|((HR_T)^m\mathrm{sgn}(c))_{k,\tilde{k}}\right|_{\mathbf{k}}^{2K}\right)\right]\\
&\leq \sum_{t=1}^{\min\{mK,N\}} \frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2mK,t)} \sum_{\substack{(k_1^{(1)},\tilde{k}_1^{(1)}),\ldots,(k_m^{(1)},\tilde{k}_m^{(1)})\in T\\ \vdots\\ (k_1^{(2K)},\tilde{k}_1^{(2K)}),\ldots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\in T\\ k_{j-1}^{(p)}\neq k_j^{(p)},\tilde{k}_{j-1}^{(p)}\neq\tilde{k}_j^{(p)},\ j\in[m]}} \times\\
&\prod_{A\subset\mathcal{A}}\left[\delta\left(\sum_{(r,p)\in A}(-1)^p\left(k_r^{(p)}-k_{r-1}^{(p)}\right)\right)+\delta\left(\sum_{(r,p)\in A}(-1)^p\left(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)}\right)\right)\right]
\end{aligned}$$

with $(k_0^{(p)},\tilde{k}_0^{(p)}) := (k,\tilde{k}), p = 1,\ldots,2K$. Hereby, we identified the partitions of $[2mK]$ in $P(2mK,t)$ with partitions of $[2K]\times[m]$ in $\mathcal{A}$.

Proof. Again, we recall the idempotent decomposition

$$\begin{aligned}
\left|((HR_T)^m\mathrm{sgnc})_{k\tilde{k}}\right|_{\mathbf{k}}^{2K} &= \left|((H_1R_T)^m\,\mathrm{sgnc}_1)_k\,\mathbf{e}^+ + ((H_2R_T)^m\,\mathrm{sgnc}_2)_{\tilde{k}}\,\mathbf{e}^-\right|_{\mathbf{k}}^{2K}\\
&= \left(\left|((H_1R_T)^m\,\mathrm{sgnc}_1)_k\right|\mathbf{e}^+ + \left|((H_2R_T)^m\,\mathrm{sgnc}_2)_{\tilde{k}}\right|\mathbf{e}^-\right)^{2K}\\
&= \left|((H_1R_T)^m\,\mathrm{sgnc}_1)_k\right|^{2K}\mathbf{e}^+ + \left|((H_2R_T)^m\,\mathrm{sgnc}_2)_{\tilde{k}}\right|^{2K}\mathbf{e}^-.
\end{aligned} \tag{1.47}$$

As the treatment of both terms in the idempotent decomposition (1.47) is identical we will omit the second case.

Take $\sigma_1 := \operatorname{sgn} c_1$. For the first term we have

$$((H_1R_T)^m\,\sigma_1)_k =$$

$$(-1)^m \sum_{j_1,\dots,j_m=1}^{N} \sum_{\substack{k_1,k_2,\dots,k_m\in T_1\\ k_{r-1}\neq k_r,\ r=1,\dots,m}} \sigma_1(k_m)\left(e^{\mathbf{i}(k_m-k_{m-1})x_{j_m}}\cdots e^{\mathbf{i}(k_1-k_0)x_{j_1}}\right),$$

with $k_0 := k$ and $T_1 = \mathrm{Proj}_1 T$, the set of all first coordinates of T. Thus,

$$\begin{aligned}
&\left|((H_1R_T)^m\sigma_1)_k\right|^2 = \left|((H_1R_T)^m\sigma_1)_{k_0}\right|^2\\
&= \sum_{\substack{j_1^{(1)},\dots,j_m^{(1)}=1\\ j_1^{(2)},\dots,j_m^{(2)}=1}}^{N} \sum_{\substack{k_1^{(1)},\dots,k_m^{(1)}\in T_1\\ k_1^{(2)},,\dots,k_m^{(2)}\in T_1\\ k_{j-1}^{(p)}\neq k_j^{(p)},\ j\in[m],\ p=1,2}} \sigma_1(k_m^{(1)})\overline{\sigma_1(k_m^{(2)})}\\
&\quad\times e^{\mathbf{i}\sum_{r=1}^m\left(k_r^{1(1)}-k_{r-1}^{1(1)}\right)x_{j_r^{(1)}}}\, e^{-\mathbf{i}\sum_{r=1}^m\left(k_r^{1(2)}-k_{r-1}^{1(2)}\right)x_{j_r^{(2)}}}
\end{aligned}$$

where $k_0^{(1)} = k_0^{(2)} = k_0 = k$. For the $2K$-th power we obtain

$$\begin{aligned}
\left|((H_1R_T)^m\sigma_1)_k\right|^{2K} = &\sum_{\substack{j_1^{(1)},\dots,j_m^{(1)}=1\\ \vdots\\ j_1^{(2K)},\dots,j_m^{(2K)}=1}}^{N} \sum_{\substack{k_1^{(1)},\dots,k_m^{(1)}\in T_1\\ \vdots\\ k_1^{(2K)},,\dots,k_m^{(2K)}\in T_1\\ k_{j-1}^{(p)}\neq k_j^{(p)},\ j\in[m],\ p=1,\dots,2K}} \sigma_1(k_m^{(1)})\overline{\sigma_1(k_m^{(2)})}\times\\
&\cdots\sigma_1(k_m^{(2K-1)})\overline{\sigma_1(k_m^{(2K)})}\, e^{\mathbf{i}\left(\sum_{p=1}^{2K}(-1)^p\sum_{r=1}^m(k_r^{(p)}-k_{r-1}^{(p)})x_{j_r^{(p)}}\right)}
\end{aligned}$$

with $k_0^{(p)} = k$, for $p = 1,\dots,2K$.
From (1.47) we obtain

$$\begin{aligned}
&\mathbb{E}_X\left[2\mathrm{Re}\left(\left|((HR_T)^m\mathrm{sgn}(c))_{k,\tilde{k}}\right|_{\mathbf{k}}^{2K}\right)\right] = \mathbb{E}_X\left[\left|((H_1R_T)^m\,\mathrm{sgnc}_1)_k\right|^{2K} + \left|((H_2R_T)^m\,\mathrm{sgnc}_2)_{\tilde{k}}\right|^{2K}\right]\\
&\leq \sum_{\substack{j_1^{(1)},\dots,j_m^{(1)}=1\\ \vdots\\ j_1^{(2K)},\dots,j_m^{(2K)}=1}}^{N} \sum_{\substack{(k_1^{(1)},\tilde{k}_1^{(1)}),\dots,(k_m^{(1)},\tilde{k}_m^{(1)})\in T\\ \vdots\\ (k_1^{(2K)},\tilde{k}_1^{(2K)}),\dots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\in T\\ k_{j-1}^{(p)}\neq k_j^{(p)},\tilde{k}_{j-1}^{(p)}\neq\tilde{k}_j^{(p)}\ j\in[m],\ p=1,\dots,2K}} \times\\
&\left(\mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{p=1}^{2K}(-1)^p\sum_{r=1}^m(k_r^{(p)}-k_{r-1}^{(p)})x_{j_r^{(p)}}\right)}\right] + \mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{p=1}^{2K}(-1)^p\sum_{r=1}^m(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})y_{j_r^{(p)}}\right)}\right]\right),
\end{aligned}\tag{1.48}$$

since $|\sigma_{k,\tilde{k}}| = 1$ for all $(k,\tilde{k})\in T$.

Let us consider the expected value appearing in these sums. As in the proof of Lemma 1.2.6 we have to take into account that some of the indices $j_r^{(p)}$ might coincide. This affords to introduce some additional notation. Let $(j_r^{(p)})_{r=1,\dots,m}^{p=1,\dots,2K} \subset \{1,\dots,N\}^{2mK}$ be some vector of indices and let

$\mathcal{A} = (A_1, \ldots, A_t),\ A_i \subset \{1, \ldots, m\} \times \{1, \ldots, 2K\}$ be a corresponding partition such that (r, p) and (r', p') are contained in the same block if and only if $j_r^{(p)} = j_{r'}^{(p')}$. For some $A \in \mathcal{A}$ we may unambiguously write j_A instead of $j_r^{(p)}$ if $(r, p) \in A$. Like in (1.44), using that all are different and that the variables (x_{j_A}, y_{j_a}) are independent we may write the expectation in (1.48) as

$$\begin{aligned}
&\mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{p=1}^{2K}(-1)^p\sum_{r=1}^m (k_r^{(p)}-k_{r-1}^{(p)})x_{j_r^{(p)}}\right)}\right] + \mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{p=1}^{2K}(-1)^p\sum_{r=1}^m (\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})y_{j_r^{(p)}}\right)}\right] \\
&= \prod_{A\subset\mathcal{A}}\left(\mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{(r,p)\in A}(-1)^p(k_r^{(p)}-k_{r-1}^{(p)})x_{j_A}\right)}\right] + \mathbb{E}_X\left[e^{\mathbf{i}\left(\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})y_{j_A}\right)}\right]\right) \\
&= \prod_{A\subset\mathcal{A}}\left(\delta\left(\sum_{(r,p)\in A}(-1)^p(k_r^{(p)}-k_{r-1}^{(p)})\right) + \delta\left(\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})\right)\right).
\end{aligned}$$

Once again, if $A \subset \mathcal{A}$ contains only one element then the last expression vanishes due to $k_r^{(p)} \neq k_{r-1}^{(p)}$ or $\tilde{k}_r^{(p)} \neq \tilde{k}_{r-1}^{(p)}$. Thus, we only need to consider partitions $\mathcal{A} \in P(2mK, t)$. Now, we can rewrite the inequality in (1.48) as

$$\begin{aligned}
&\mathbb{E}_X\left[2\mathrm{Re}\left(\left|((HR_T)^m \mathrm{sgn}(c))_{k,\tilde{k}}\right|_{\mathbf{k}}^{2K}\right)\right] \\
&\leq \sum_{t=1}^{mK}\sum_{\mathcal{A}\in P(2mK,t)}\sum_{\substack{j_{(1)},\ldots,j_{(t)}=1\\ \text{all different}}}^{N}\ \sum_{\substack{(k_1^{(1)},\tilde{k}_1^{(1)}),\ldots,(k_m^{(1)},\tilde{k}_m^{(1)})\in T\\ \vdots\\ (k_1^{(2K)},\tilde{k}_1^{(2K)}),\ldots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\in T\\ k_{j-1}^{(p)}\neq k_j^{(p)},\tilde{k}_{j-1}^{(p)}\neq\tilde{k}_j^{(p)}\ j\in[m],\ p=1,\ldots,2K}} \times \\
&\quad \prod_{A\subset\mathcal{A}}\left(\delta\left(\sum_{(r,p)\in A}(-1)^p(k_r^{(p)}-k_{r-1}^{(p)})\right) + \delta\left(\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})\right)\right) \\
&= \sum_{t=1}^{\min\{mK,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2mK,t)}\ \sum_{\substack{(k_1^{(1)},\tilde{k}_1^{(1)}),\ldots,(k_m^{(1)},\tilde{k}_m^{(1)})\in T\\ \vdots\\ (k_1^{(2K)},\tilde{k}_1^{(2K)}),\ldots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\in T\\ k_{j-1}^{(p)}\neq k_j^{(p)},\tilde{k}_{j-1}^{(p)}\neq\tilde{k}_j^{(p)}\ j\in[m],\ p=1,\ldots,2K}} \times \\
&\quad \prod_{A\subset\mathcal{A}}\left(\delta\left(\sum_{(r,p)\in A}(-1)^p(k_r^{(p)}-k_{r-1}^{(p)})\right) + \delta\left(\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})\right)\right)
\end{aligned}$$

□

Again, and for future use, we shall denote

$$B(\mathcal{A},T) := \sum_{\substack{(k_1^{(1)},\tilde{k}_1^{(1)}),\dots,(k_m^{(1)},\tilde{k}_m^{(1)})\in T \\ \vdots \\ (k_1^{(2K)},\tilde{k}_1^{(2K)}),\dots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\in T \\ k_{j-1}^{(p)}\neq k_j^{(p)},\tilde{k}_{j-1}^{(p)}\neq \tilde{k}_j^{(p)}\, j\in[m],\, p=1,\dots,2K}} \times$$

$$\prod_{A\subset\mathcal{A}}\left(\delta\left(\sum_{(r,p)\in A}(-1)^p(k_r^{(p)}-k_{r-1}^{(p)})\right)+\delta\left(\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)}-\tilde{k}_{r-1}^{(p)})\right)\right).$$

1.2.6 Proof of Main Theorems

Proof of Theorem 1.2.1

The goal is to complete the proof of Theorem 1.2.1 with the help of Lemma 1.2.6 and subsequent results.

Proof. First, we take a closer look into expression (1.46) for $C(\mathcal{A},T)$, where $\mathcal{A}\in P(2n,t)$. Here, the indices $\Big((k_1,\tilde{k}_1),\dots,(k_{2n},\tilde{k}_{2n})\Big)\in T_1^{2n}\times T_2^{2n}$ are subjected to the $|\mathcal{A}|=t$ linear constraints $\sum_{r\in A}(k_{r+1}-k_r)=0$ and $\sum_{r\in A}\Big(\tilde{k}_{r+1}-\tilde{k}_r\Big)=0$ for all $A\in\mathcal{A}$. These constraints are independent except for $\sum_{r=1}^{2n}(k_{r+1}-k_r)=0$ and $\sum_{r=1}^{2n}\Big(\tilde{k}_{r+1}-\tilde{k}_r\Big)=0$. Thus, we get for (1.46) the estimate

$$C(\mathcal{A},T)\leq |T|^{2n-t+1}\leq M^{2n-t+1}, \tag{1.49}$$

in terms of the sparsity M. We remark that in Theorem 1.2.1 T is not random, so that $\mathbb{E}=\mathbb{E}_X$. By Lemma 1.2.6 we obtain

$$\begin{aligned}\mathbb{E}\left[\|H_0^n\|_F^2\right] &\leq \sum_{t=1}^{\min\{n,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2n,t)}|T|^{2n-t+1}\\ &\leq M^{2n+1}\sum_{t=1}^{n}\left(\frac{N}{M}\right)^t S_2(2n,t),\end{aligned}$$

where $S_2(n,t)=|P(2n,t)|$ are the associated Stirling numbers of the second kind. We define the auxiliar function $F_n, n\in\mathbb{N}$, by

$$F_n(\theta):=\sum_{k=1}^{[n/2]}S_2(n,k)\theta^k,$$

and we set $\theta=\frac{N}{M}$.

$$\begin{aligned}\mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\|_F\geq\kappa\right) &= \mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\|_F^2\geq\kappa^2\right)\leq\kappa^{-2}\,\mathbb{E}\left[\left\|\left(N^{-1}H_0\right)^n\right\|_F^2\right]\\ &\leq \kappa^{-2}\,M\,\theta^{-2n}F_{2n}(\theta)=\kappa^{-2}\;M\,G_{2n}(\theta).\end{aligned}$$

So, $\left\| \left(N^{-1}H_0\right)^n \right\|_F \leq \kappa < 1$ implies that $\left(I - \left(N^{-1}H_0\right)^n\right)$ is invertible by the von Neumann series and, therefore, also $[\mathcal{F}_{TX}^* \mathcal{F}_{TX}] = N\left(I - N^{-1}H_0\right)$ is invertible. In particular, we conclude that $\mathcal{F}_{TX}$ is injective. This proves that the injectivity condition in Lemma (1.2.4) is satisfied automatically with a probability that can be derived from the estimation above.

Now, we turn our attention to the terms $\mathbb{P}(E_{k\tilde{k}})$. By Lemma 1.2.7 we need to bound $B(\mathcal{A}, T)$, i.e., the number of vectors $\left(k_j^{(p)}, \tilde{k}_j^{(p)}\right) \in T^{2mK}$ satisfying $\sum_{(r,p)\in A}(-1)^p(k_r^{(p)} - k_{r-1}^{(p)}) = 0$ and $\sum_{(r,p)\in A}(-1)^p(\tilde{k}_r^{(p)} - \tilde{k}_{r-1}^{(p)}) = 0$ for all $A \in \mathcal{A}$ with $\mathcal{A} \in P(2mK, t)$. As these are t independent linear constraints the number of these indices is bounded from above by $|T|^{2mK-t} \leq M^{2mK-t}$. Thus, taking $\theta = \frac{N}{M}$, we obtain (similar as above)

$$\begin{aligned}
\mathbb{E}\left[\left|\left(\left(N^{-1}HR_T\right)^m \operatorname{sgn} c\right)_{k\tilde{k}}\right|^{2K}\right] &\leq \sum_{t=1}^{mK} N^t S_2(2mK, t) M^{2mK-t} \\
&= M^{2mK} F_{2mK}(\theta). \qquad (1.50)
\end{aligned}$$

Now, let $K_1, \cdots, K_n$ be natural numbers and choose $\beta > 0$ such that $\sum_{m=1}^n \beta^{n/K_m} = a_1$ (where a_1 is as in 1.34). Then, for every $(k, \tilde{k}) \in T$ it holds for $\mathbb{P}(E_{k,\tilde{k}})$, see (1.35), that

$$\begin{aligned}
\mathbb{P}(E_{k,\tilde{k}}) &= \mathbb{P}\left(\sum_{m=1}^n \left|\left(\left(N^{-1}HR_T\right)^m \operatorname{sgn} c\right)_{k,\tilde{k}}\right| \geq a_1\right) \\
&\leq \sum_{m=1}^n \mathbb{P}\left(N^{-m}\left|\left((HR_T)^m \operatorname{sgn} c\right)_{k,\tilde{k}}\right| \geq \beta^{n/K_m}\right) \\
&= \sum_{m=1}^n \mathbb{P}\left(N^{-2mK_m}\left|\left((HR_T)^m \operatorname{sgn} c\right)_{k,\tilde{k}}\right|^{2K_m} \geq \beta^{2n}\right).
\end{aligned}$$

Since $\mathbb{P}(|x| \geq \gamma) = \mathbb{P}(\frac{1}{\gamma}|x| \geq 1) \leq \mathbb{E}[\frac{1}{\gamma}|x|] = \frac{1}{\gamma}\mathbb{E}[|x|], (\gamma > 0)$, we conclude

$$\begin{aligned}
\mathbb{P}(E_{k,\tilde{k}}) &\leq \sum_{m=1}^n \mathbb{P}\left(N^{-2mK_m}\left|\left((HR_T)^m \operatorname{sgn} c\right)_{k,\tilde{k}}\right|^{2K_m} \geq \beta^{2n}\right) \\
&\leq \beta^{-2n} \sum_{m=1}^n N^{-2mK_m} \mathbb{E}\left[\left|\left((HR_T)^m \operatorname{sgn} c\right)_{k,\tilde{k}}\right|^{2K_m}\right]. \qquad (1.51)
\end{aligned}$$

From (1.50) we finally obtain

$$\mathbb{P}(E_{k,\tilde{k}}) \leq \beta^{-2n} \sum_{m=1}^n \left(\frac{M}{N}\right)^{2mK_m} F_{2mK_m}\left(\frac{N}{M}\right) = \beta^{-2n} \sum_{m=1}^n \theta^{-2mK_m} F_{2mK_m}(\theta). \qquad (1.52)$$

Let $\mathbb{P}$(failure) denote the probability that an exact reconstruction of f by ℓ_1-minimization fails.

By Lemma 1.2.6, it follows

$$\mathbb{P}(\text{failure}) \leq \mathbb{P}\left(\{\mathcal{F}_{TX} \text{ is not injective}\} \cup \left\{\sup_{(k,\tilde{k})\in T^c} |P_{k\tilde{k}}| \geq 1\right\}\right)$$

$$\leq \sum_{(k,\tilde{k})\in\mathbb{Z}_\rho^2} \mathbb{P}(E_{k\tilde{k}}) + \mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\| \geq \kappa\right) \leq\ D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2}\ M\, G_{2n}(\theta)$$

under the conditions $a := a_1 \sum_{m=1}^{n} \beta^{n/K_m} < 1,\ a_2 = 1-a$, and

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1} M^{-3/2} = \frac{1-a}{1+a} M^{-3/2}.$$

□

Let us remarks that, given n, it is reasonable to take $K_m \approx m/n,\ m = 1,\ldots,n$. Then, β is chosen quite close to the maximal value such that $a_1 = \sum_{m=1}^{n} \beta^{n/K_m} < 1$. By our choice of K_m we approximately get

$$\sum_{m=1}^{n} \beta^{n/K_m} \approx \sum_{m=1}^{n} \beta^m \approx \frac{\beta}{1-\beta}.$$

Thus, the optimal β will always be close to $1/2$.

Proof of Theorem 1.2.2

The proof of the Theorem 1.2.2 is the same as in [69], due to the fact that it depends only on the set partition and it does not enter into account with the algebraic structure of the bicomplex numbers. Therefore, the proof of this theorem is essentially the same as in as [69]. For the sake of convenience for the reader we present it in this subsection.

Proof. We will refine the probability bound (1.23) of Theorem 1.2.1 in order to obtain Theorem 1.2.2. First, we proof that the associated Stirling numbers satisfy the estimate

$$S_2(n,k) \leq (3n/2)^{n-k} \quad \text{for all } k = 1,\ldots,\lfloor n/2 \rfloor. \tag{1.53}$$

Indeed, $S_2(1,k) = 0$ and $S_2(2,1) = 1$. By induction over n and using the recursion formula ((2.3) in Set Partitions), namely

$$S_2(n,k) = kS_2(n-1,k) + (n-1)S_2(n-1,k-1),$$

it follows

$$\begin{aligned} S_2(n,k) &= kS_2(n-1,k) + (n-1)S_2(n-2,k-1) \\ &\leq k(3(n-1)/2)^{n-k-1} + (n-1)(3n/2-3)^{n-k-1} \leq (n-1+k)(3n/2)^{n-k-1} \\ &\leq (3n/2)^{n-k}, \end{aligned}$$

since $(n-1+k) \leq 3n/2$. Plugging (1.53) into the definition of G_{2n} yields

$$G_{2n}(\theta) = \theta^{-2n}\sum_{k=1}^{n} S_2(2n,k)\theta^k \leq \theta^{-2n}\sum_{k=1}^{n}(3n)^{2n-k}\theta^k = (3n/\theta)^{2n}\sum_{k=1}^{n}(\theta/3n)^k$$

$$
\begin{aligned}
&= (3n/\theta)^{2n-1}\frac{1-(\theta/3n)^n}{1-(\theta/3n)} = (3n/\theta)^n\frac{1-(3n/\theta)^n}{1-3n/\theta} \\
&\leq \frac{\delta^n}{1-\delta},
\end{aligned}
$$

for n chosen in such way that $3n/\theta \leq \delta < 1$. Now consider the term $D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}$ from the probability bound (1.23). We choose $K_m = r(n/m)$ where $r(x)$ denotes the function that rounds x to the nearest integer. Then,

$$
mK_m \in \{\lceil 2n/3\rceil, \ldots, \lfloor 4n/3\rfloor\}, \quad m \in \{1, \ldots, n\},
$$

and, thus,

$$
\sum_{k=1}^{n} G_{2mK_m}(\theta) \leq \max_{k\in\{\lceil 2n/3\rceil,\ldots,\lfloor 4n/3\rfloor\}} G_{2k}(\theta) \leq \max_{k\in\{\lceil 2n/3\rceil,\ldots,\lfloor 4n/3\rfloor\}} \frac{\delta^k}{1-\delta} \leq n\frac{\delta^{2n/3}}{1-\delta}
$$

provided $3k/\theta \leq \delta$ for all $k \in \{\lceil 2n/3\rceil, \ldots, \lfloor 4n/3\rfloor\}$. This yields

$$
D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}(\theta) \leq Dn\frac{1}{1-\delta}\left(\beta^{-3}\delta\right)^{2n/3}.
$$

In order to make this expression small enough we choose $\delta := \beta^3 e^{-3\tau/2}$, for some $\tau > 0$. Then,

$$
D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}(\theta) \leq \frac{Dn}{1-\delta}e^{-n\tau}.
$$

The last term in this inequality is smaller than $\epsilon/2$ if

$$
\ln(D/\epsilon) \leq n\tau + \ln\left(\frac{1-\delta}{2n}\right).
$$

Let us assume $n \geq S$. Then,

$$
n\tau + \ln\left(\frac{1-\delta}{2n}\right) \geq n\left(\tau + \frac{\ln\left(\frac{1-\delta}{2S}\right)}{S}\right).
$$

Since $3n/\theta \leq \delta$, we choose now

$$
n = \lfloor \delta\theta/4 \rfloor \geq \frac{S-1}{S}\delta\theta/4 = \frac{S-1}{4S}\beta^3 e^{-3\tau/2}\theta, \tag{1.54}
$$

and for those chosen values, we obtain a constant $Q(\beta, \tau, S)$ such that

$$
n\tau + \ln\left(\frac{1-\delta}{2n}\right) \geq Q(\beta, \tau, S)\theta.
$$

Moreover,

$$D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}(\theta) \le \epsilon/2 \quad \text{if} \quad \theta \ge \frac{\ln(D/\epsilon)}{Q(\beta,\tau,S)}$$

Recalling that $\theta = M/N$ it follows that there exists a constant C_1 such that $D\beta^{-2n}\sum_{m=1}^{n} G_{2mK_m}(\theta) \le \epsilon/2$ provided

$$N \ge C_1 M \ln(D/\epsilon).$$

Now consider the other term $M\kappa^{-2}G_{2n}(\theta)$ in the probability bound (1.23). We choose κ such that

$$\kappa = \frac{(1-a)/(1-a)M^{-3/2}}{1+(1-a)/(1+a)M^{-3/2}} \ge \frac{1-a}{2(1+a)}M^{-3/2}.$$

Hence,

$$M\kappa^{-2}G_{2n}(\theta) \le \left(\frac{1-a}{2(1+a)}\right)^2 M^4 G_{2n}(\theta).$$

Now we do not have the freedom anymore to choose n. We have to make the same choice as in (1.54). This yields

$$M\kappa^{-2}G_{2n}(\theta) \le \left(\frac{2(1+a)}{(1-a)}\right)^2 M^4 \left(\frac{3\beta^3}{8}\right)^{n(\theta)-1}.$$

Requiring that the latter expression is less than $\epsilon/2$ is equivalent to

$$(n(\theta)-1)\ln\left(\frac{8}{3\beta^3}\right) \ge \ln\left(8\left(\frac{1+a}{1-a}\right)^2\right) + 4\ln(M) + \ln(\epsilon^{-1}).$$

As in [69] a simple numerical test shows that a valid choice for $\beta = 0.47$. This results in $a \le 0.957$ and $n(\theta) \approx \lfloor 0.013\theta \rfloor$. Hence, $\ln(8/(3\beta^3)) \approx 3.2459$ and $\ln(8((1+a)/(1-a))^2) \approx 9.7153$. Since $M \le D$ there exists a constant C_2 (whose precise value may be calculated from the numbers above) such that $M\kappa^{-2}G_{2n}(\theta) \le \epsilon/2$ provided

$$N \ge C_2 M(\ln(D) + \ln(\epsilon^{-1})).$$

Choosing $C := \max\{C_1, C_2\}$ completes the proof of Theorem 1.2.2. $\square$

Chapter 2

Quaternionic signal

> *Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.*

David Hilbert,

Another generalization of complex numbers to higher dimensions is given by the algebra of quaternions. This algebra was introduced by Hamilton in 1848 [44]. Although later being overpassed in favor of Gibb's calculus, quaternions gained a new life due to its applicability in signal processing. This is due to both, the applicability of quaternion-valued functions to color-coded images as well as the link to new concepts of higher-dimensional phases, like the hypercomplex signal of Bülow or the monogenic signal by Larkin and Felsberg. Here we want to apply the same scheme as in the previous chapter to the case of quaternion-valued functions. The main obstacle in the study of quaternion-valued matrices and functions, as expected, comes from the non-commutative nature of quaternionic multiplication.

2.1 Quaternionic Linear Algebra and Quaternionic Analysis

2.1.1 Definitions

The algebra of quaternions $\mathbb{H}$ is a four-dimensional real associative division algebra with unit 1 spanned by the elements $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ endowed with the relations

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -1, \quad \mathbf{IJ} = -\mathbf{JI} = \mathbf{K}.$$

This algebra is non-commutative. The real and imaginary parts of a given quaternion

$$q = x_0 1 + x_1 \mathbf{I} + x_2 \mathbf{J} + x_3 \mathbf{K}$$

are defined as $\mathrm{Re}(q) = q_0 := x_0$, and $\mathrm{Im}(q) := x_1 \mathbf{I} + x_2 \mathbf{J} + x_3 \mathbf{K}$. Therefore, we have the natural embeddings of the real numbers and of $\mathbb{R}^3$ into quaternions given by

$$x_0 \in \mathbb{R} \to x_0 1 \in \mathbb{H} \quad \text{and} \quad (x_1, x_2, x_3) \in \mathbb{R}^3 \to x_1\mathbf{I} + x_2\mathbf{J} + x_3\mathbf{K} \in \mathbb{H}.$$

This leads to the identifications

$$\mathbb{H} \equiv \mathbb{R}^4, \quad \operatorname{Im}\mathbb{H} \equiv \mathbb{R}^3, \quad \operatorname{Re}\mathbb{H} \equiv \mathbb{R},$$

where $\operatorname{Im}\mathbb{H}$ is the three dimensional space of pure imaginary quaternions, and hence, $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$.

The *conjugation* on $\mathbb{H}$ is an automorphism of $\mathbb{H}$ onto itself given by

$$q = x_0 + \operatorname{Im} q \quad \to \quad \overline{q} = x_0 - \operatorname{Im} q,$$

together with to the involution property

$$\overline{qp} = \overline{p}\,\overline{q}, \ \forall p, q \in \mathbb{H}.$$

A purely imaginary quaternion with absolute value 1 is called an imaginary unit. We denote the set of all imaginary units by $\mathbb{S}^2$, that is,

$$\mathbb{S}^2 = \left\{ x_1\mathbf{I} + x_2\mathbf{J} + x_3\mathbf{K} \in \mathbb{H} \, : \, \sum_{i=1}^{3} x_i^2 = 1 \right\}.$$

2.1.2 Inner and outer products

Given two quaternions $q, p \in \mathbb{H}$ its quaternionic multiplication can be expressed in terms of the usual scalar and vector products on $\operatorname{Im}\mathbb{H} \sim \mathbb{R}^3$ by

$$q\overline{p} = (q_0 + \operatorname{Im} q)(p_0 - \operatorname{Im} p) = \underbrace{q_0 p_0 + \operatorname{Im} q \cdot \operatorname{Im} p}_{\in \mathbb{R}} + \underbrace{(-q_0 \operatorname{Im} p + p_0 \operatorname{Im} q - \operatorname{Im} q \times \operatorname{Im} p)}_{\in \operatorname{Im}\mathbb{H}}.$$

Moreover, we have

$$\langle q, p \rangle = \operatorname{Re}(q\overline{p}),$$

where $\langle \cdot, \cdot \rangle$ corresponds to the Euclidean scalar product defined on $\mathbb{H} \sim \mathbb{R}^4$, and the induced norm

$$\|q\|^2 = \operatorname{Re}(q\overline{q}) = \langle q, q \rangle$$

which satisfies the product rule $\|qp\| = \|q\| \, \|p\|$. Also, for every non-zero quaternion q it holds

$$q^{-1} = \frac{\overline{q}}{\|q\|^2}.$$

Furthermore, we can introduce the signum of a quaternion q defined via

$$\operatorname{sgn} q = \begin{cases} \frac{q}{\|q\|}, & q \neq 0 \\ 0, & q = 0. \end{cases}$$

Note that $\operatorname{sgn} q \in \mathbb{H}$. For the $\operatorname{sgn} q$ we have the following properties.

Lemma 2.1.1. *Let* $q = q_0 + q_1\mathbf{I} + q_2\mathbf{J} + q_3\mathbf{IJ}$ *and* $p = p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}$ *be two quaternions. It holds*

1. $\|\operatorname{sgn} p\| = 1$, $p \neq 0$*;*
2. $\overline{\operatorname{sgn} p}\operatorname{sgn} p = 1$, $p \neq 0$.
3. $p\,\overline{\operatorname{sgn} p} = \|p\|$*;*
4. $\|q\|\,\|\operatorname{sgn} p\| = \|q \operatorname{sgn} p\|$;

Proof. Without loss of generality we assume $p \neq 0$. Let us recall $\|\operatorname{sgn} p\| = \|\overline{\operatorname{sgn} p}\|$ and start with 1..

$$\|\operatorname{sgn} p\| = \left\| \frac{p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} \right\| = \frac{\|p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}\|}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} = 1.$$

The second item 2. can be shown simply by

$$\overline{\operatorname{sgn} p}\operatorname{sgn} p = \frac{p_0 - p_1\mathbf{I} - p_2\mathbf{J} - p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} \frac{p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} = \frac{\bar{p}p}{\|p\|^2} = 1.$$

For 3. we have

$$p\,\overline{\operatorname{sgn} p} = (p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ})\frac{p_0 - p_1\mathbf{I} - p_2\mathbf{J} - p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} = \frac{p_0^2 + p_1^2 + p_2^2 + p_3^2}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} = \|p\|$$

and, finally, for 4. we obtain

$$\begin{aligned} \|q \operatorname{sgn} p\| &= \left\| (q_0 + q_1\mathbf{I} + q_2\mathbf{J} + q_3\mathbf{IJ})\frac{p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} \right\| \\ &= \|q_0 + q_1\mathbf{I} + q_2\mathbf{J} + q_3\mathbf{IJ}\| \left\| \frac{p_0 + p_1\mathbf{I} + p_2\mathbf{J} + p_3\mathbf{IJ}}{\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}} \right\| \\ &= \|q_0 + q_1\mathbf{I} + q_2\mathbf{J} + q_3\mathbf{IJ}\| = \|q\| = \|q\|\,\|\operatorname{sgn} p\|. \end{aligned}$$

Note that the property 1. means that $\operatorname{sgn} q \in \mathbb{S}^3 = \left\{ x_0 + x_1\mathbf{I} + x_2\mathbf{J} + x_3\mathbf{K} \in \mathbb{H} \,:\, \sum_{i=0}^3 x_i^2 = 1 \right\}$. In 4. we still have for quaternions the preservation of the modulus $\|p\,q\| = \|p\|\|q\|$ which is not true in higher dimensional analogues, like Clifford algebras. □

2.1.3 Quaternionic exponentials

Lemma 2.1.2. *Let p be an invertible quaternion. Then, the map $R_p(q) = p\,q\,p^{-1}$ is an orthonormal map from $\operatorname{Im}\mathbb{H} \sim \mathbb{R}^3$ onto itself.*

Proof. First, observe that if $q \in \operatorname{Im}\mathbb{H}$ then $q = \operatorname{Im} q$ and

$$p(\operatorname{Im} q)\overline{p} = p_0^2 \operatorname{Im} q + 2p_0 \operatorname{Im} p \times \operatorname{Im} q - (\operatorname{Im} p \cdot \operatorname{Im} q)\operatorname{Im} p + (\operatorname{Im} p \times \operatorname{Im} q) \times \operatorname{Im} p \in \operatorname{Im}\mathbb{H}.$$

Finally, the equality

$$\|R_p(q)\|^2 = \|p\,q\,p^{-1}\|^2 = \|q\|^2,$$

completes the proof. □

Theorem 2.1.3. *Let q be an invertible quaternion. Then,*

$$q = \|q\|\left[\cos(\theta) + \omega(q)\sin(\theta)\right],$$

where $\omega(q) = \frac{\operatorname{Im} q}{\|\operatorname{Im} q\|} \in \mathbb{S}^2 \subset \operatorname{Im}\mathbb{H}$ and $\theta \in [0,\pi]$ is such that $\cos\theta = \frac{q_0}{\|q\|}$ and $\sin\theta = \frac{\|\operatorname{Im} q\|}{\|q\|}$.

Proof. Immediate, since

$$q = \|q\|\left(\frac{q_0}{\|q\|} + \frac{\operatorname{Im} q}{\|q\|}\right) = \|q\|\left(\frac{q_0}{\|q\|} + \frac{\operatorname{Im} q}{\|\operatorname{Im} q\|}\frac{\|\operatorname{Im} q\|}{\|q\|}\right).$$

and setting $\cos\theta = \frac{q_0}{\|q\|}$ leads to

$$\sin^2\theta = 1 - \frac{q_0^2}{\|q\|^2} = \frac{\|\operatorname{Im} q\|^2}{\|q\|^2},$$

with $\omega(q) := \frac{\operatorname{Im} q}{\|\operatorname{Im} q\|} \in \mathbb{S}^2$ the unit sphere in $\operatorname{Im}\mathbb{H}$. □

This leads to the following definition (for instance, see [41]):

Definition 2.1.4. *Given a purely imaginary quaternion $\omega \in \mathbb{S}^2 \subset \operatorname{Im}\mathbb{H}$, we define the quaternionic exponential $e^{\omega\theta}$ as*

$$\theta \in \mathbb{R} \mapsto e^{\omega\theta} := \cos(\theta) + \omega\sin(\theta) \in \mathbb{H}. \tag{2.1}$$

We list some of its properties.

- $\|e^{\omega\theta}\| = 1$;
- $\|e^{\omega\theta}\| \le e^{|\theta|}$;
- given $\omega_1, \omega_2 \in \mathbb{S}^2$, we have

  $$e^{\omega_1\theta_1}e^{\omega_2\theta_2} = e^{\omega_1\theta_1 + \omega_2\theta_2}, \quad \forall \theta_1, \theta_2,$$

  only if $\omega_1\omega_2 = \omega_2\omega_1$.

The last property shows the non-commutative character of quaternionic exponentials and one of the major problems in the application of the scheme from the previous chapter.

2.2 Sparse sampling of quaternionic signals

2.2.1 The setting

While there is quite a variety of possibilities to consider concrete quaternionic Fourier transforms [73] here we will consider decompositions with respect to the atoms $\Psi(x,y) = e^{\mathbf{I}kx}e^{\mathbf{J}\tilde{k}y}$. This corresponds to the main applications in image processing (c.f. [14]) and can also be easily adapted to the other cases. As in the previous chapter we denote by $\prod_\rho$ the space of all quaternionic trigonometric polynomials of maximal order $\rho \in \mathbb{N}_0$ in two real variables. The dimension of $\prod_\rho$ is $D := (2\rho+1)^2$. Therefore, an element $q \in \prod_\rho$ is of the form

$$q(x,y) = \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} e^{\mathbf{I}kx}e^{\mathbf{J}\tilde{k}y}c_{k,\tilde{k}}, \qquad (x,y)\in[0,2\pi]^2,\ c_{k,\tilde{k}}\in\mathbb{H}, \tag{2.2}$$

where **I** and **J** the basis elements of $\mathbb{H}$. We want to determine the coefficients from a few given samples. To this end we consider a sequence of coefficients $c := (c_{k,\tilde{k}})_{k,\tilde{k}}$ as a vector such that c is supported on a set T which has a much smaller cardinality than the dimension of $\prod_\rho$. That is to say, the finite combination in (2.2) is "sparse". However, we do not require any information on T except its maximum size. Therefore, we introduce the set (not a linear space) $\prod_\rho(M) \subset \prod_\rho$ of all polynomials of type (2.2) such that the sequence of their coefficients c has support on a set $T \subset [-\rho,\rho]^2 \cap \mathbb{Z}^2$ with $|T| \le M$, i.e., $q \in \prod_\rho(M)$ is of the form

$$q(x,y) = \sum_{(k,\tilde{k})\in T\subset[-\rho,\rho]^2\cap\mathbb{Z}^2} e^{\mathbf{I}kx}e^{\mathbf{J}\tilde{k}y}c_{k,\tilde{k}}.$$

Furthermore, we consider the given sampling set $X := \{(x_1,y_1),(x_2,y_2),\dots,(x_N,x_N)\}$ as a set of independent random variables having uniform distribution on $[0,2\pi]^2$, with $(x_i,y_i)\in\mathbb{R}^2$ belonging to the grid. Thus, the main objective is to reconstruct $q \in \prod_\rho(M)$ from the samples $f(x_i,y_i)$ at those N randomly chosen points. We join them into the sampling set

$$X := \{(x_1,y_1),(x_2,y_2),\dots,(x_N,y_N)\},$$

where the cardinality $|X| = N$, which N is the number of given samples.

The standard way of obtaining the coefficients $c_{k,\tilde{k}}$ would be by applying a greedy algorithm, such as ℓ_0-minimization or matching pursuit. It basically corresponds to searching for the atom with the largest coefficient in each step ([81]). Unfortunately, such an algorithm is NP-hard with exponentially growing computational costs. Therefore, we would like to use a different algorithm, such as basis pursuit. This consists in the following non-linear method of reconstructing $q \in \prod_\rho(M)$ from its sampled values $(q(x_1,y_1),\dots,q(x_N,y_N))$. We minimize the ℓ_1-norm of the

Fourier coefficients $c_{k,\tilde{k}}$,

$$||(c_{k,\tilde{k}})||_1 := \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} |c_{k,\tilde{k}}|,$$

under the constraint that the corresponding trigonometric polynomial matches q on the sampling points. In other words we solve the problem

$$||(c_{k,\tilde{k}})||_1 \qquad \text{s.t.} \qquad \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} e^{\mathbf{I}kx_i} e^{\mathbf{J}\tilde{k}y_i} c_{k,\tilde{k}} = q(x_i, y_i), \ \ i = 1, \dots, N. \tag{2.3}$$

We can directly point out that *basis pursuit* [22] can be performed with efficient convex optimization techniques [12] via linear programming. This makes it a much faster algorithm than the simple ℓ^0-minimization. But, of course, the immediate question is if we indeed can replace the ℓ_0-minimization by an ℓ_1-minimization. This question will be answered in the next section.

2.2.2 Equivalence between ℓ_0-minimization and basis pursuit

Similar to the bicomplex case we would like to use *basis pursuit* [22] to reconstruct f completely by determining all the coefficients $c_{k\tilde{k}}$, $(k,\tilde{k}) \in [-\rho,\rho]^2 \cap \mathbb{Z}^2$. Unfortunately, a condition like the restrictive isometry property (RIP, [6]) which is valid in all cases (including the worst) is too restrictive for most applications. Therefore, we are going to give a probabilistic answer. The theorems below are analogues to the theorems given in Candes, Romberg, and Tao [19] and in Rauhut [69]. In terms of the position of the sampling points, our results apply to sampling points taken randomly, but with uniform distribution, from the grid.

Let us start with the following theorems which will be proved in parallel in the sub-section 2.2.3.

Theorem 2.2.1. *Assume $f \in \prod_\rho(M)$ with some sparsity $M \in \mathbb{N}$. Let $(x_1,y_1),\dots,(x_N,y_N) \in [0,2\pi]^2 \cap \mathbb{Z}^2$ be independent random variables having uniform distribution on $[0,2\pi]^2$. Choose $n \in \mathbb{N}$, $\beta > 0$, $\kappa > 0$ and $K_1,\cdots,K_n \in \mathbb{N}$ such that*

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \ \text{ and } \ \frac{\kappa}{1-\kappa} \leq \frac{1-a}{1+a} M^{-3/2}. \tag{2.4}$$

Set $\theta := N/M$ and set $D = (2\rho+1)^2$. Then with probability at least

$$1 - \left(D\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2}\, M\, G_{2n}(\theta) \right) \tag{2.5}$$

f can be reconstructed exactly from its sample values $f(x_1,y_1),\dots,f(x_N,y_N)$ by solving the ℓ_1-minimization problem

$$\min \left\| (c_{k,\tilde{k}}) \right\|_1 := \min \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} |c_{k,\tilde{k}}|,$$

$$s.t. \quad f(x_i, y_i) := \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} e^{\mathbf{I}kx_i} e^{\mathbf{J}\tilde{k}y_i} c_{k,\tilde{k}}, \quad i = 1, \ldots, N. \tag{2.6}$$

While the above theorem provides exact constants we can give a version of the theorem which is somewhat easier to apply.

Theorem 2.2.2. *There exists an absolute constant $C > 0$ such that the following is true. Assume $f \in \prod_\rho(M)$ for some sparsity $M \in \mathbb{N}$. Let $(x_1, y_1), \ldots, (x_N, y_N) \in [0, 2\pi]^2$ be independent random variables having the uniform distribution on $[0, 2\pi]^2$. If for some $\epsilon > 0$ it holds*

$$N \geq CM \log(D/\epsilon) \tag{2.7}$$

then with probability at least $1 - \epsilon$ the function f can be recovered from its sample values $f(x_i, y_i)$, $i = 1, \ldots, N$, by solving the minimization problem (2.6).

We will announce our third theorem about reconstructing a sparse trigonometric polynomial from random samples. So, for convenience, we will use the notation $\prod_T$ as the set of all trigonometric polynomials whose coefficients are supported on T.

For the next Theorem 2.2.3 we model the set $T \subset [\rho, \rho]^2$ of non-vanishing Fourier coefficients as random. Here we will treat the "generic" case and not the arbitrary sparse polynomials. We expect so better estimates for the probability of exact reconstruction.

Consider $0 < \tau < 1$. The probability that an index $(k, \tilde{k}) \in [\rho, \rho]^2$ belongs to T is modelled as

$$\mathbb{P}((k, \tilde{k}) \in T) = \tau \tag{2.8}$$

independently of each $(k, \tilde{k})$. The choice of the sampling set X are stochastically independent. Here we also assume that choice of T. So the length of T, i.e., $|T|$ proceed the binomial distribution and the expected size of $|T|$ is, obviously, $\mathbb{E}|T| = \tau D = \tau(2\rho + 1)^2$.

We will define the two auxiliary notation before to announce the third theorem. For $n \in \mathbb{N}$ we define

$$W(n, N, \mathbb{E}|T|, D) := N^{-2n} \sum_{t=1}^{\min\{n,N\}} \frac{N!}{(N-t)!} \sum_{s=2}^{2n} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} Q(2n, t, s, R) D^{-R} \tag{2.9}$$

and for $K, m \in \mathbb{N}$

$$Z(K, m, N, \mathbb{E}|T|, D) := N^{-2Km} \sum_{t=1}^{\min\{Km,N\}} \frac{N!}{(N-t)!} \sum_{s=1}^{2Km} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}} Q^*(2K, m, t, s, R) D^{-R}. \tag{2.10}$$

By using Basis Pursuit our theorem is given as follows.

Theorem 2.2.3. *Let $(x_1, y_1), \ldots, (x_N, y_N) \in [0, 2\pi]^2$ be independent random variables having the uniform distribution on $[0, 2\pi]^2$. Further assume that $T := T_1 \times T_2$ is (which is an independent* *set of* $(x_{i_1}, y_{i_1}), \ldots, (x_{i_N}, y_{i_N}))$ *random subset of modelled by* $\mathbb{P}((k,\tilde{k}) \in T) = \tau$ *such that* $\mathbb{E}|T| = \tau D \geq 1$. *Choose* $n \in \mathbb{N}$, $\alpha, \beta > 0$ *and* $K_1, \ldots, K_n \in \mathbb{N}$ *such that*

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \qquad \text{and} \qquad \frac{k}{1-k} \leq \frac{1-a}{1+a}((\alpha+1)\mathbb{E}|T|)^{-3/2}. \tag{2.11}$$

Then with probability at least

$$1 - \left(\kappa^{-2} W(n, N, \mathbb{E}|T|, D) + \beta^{-2n} D \sum_{m=1}^{n} Z(K_m, m, N, \mathbb{E}|T|, D) + \exp\big(-\frac{3\alpha^2}{6+2\alpha}\mathbb{E}|T|\big)\right) \tag{2.12}$$

any $f \in \prod_T \subset \prod_\rho(|T|)$ *can be reconstructed exactly from its sample values* $f(x_1, y_1), \ldots, f(x_N, y_N)$ *by solving the minimization problem (2.6).*

In the next section we will give the proof of these theorems.

2.2.3 Proof of the main theorems

First of all, we need to introduce some auxiliary notation. We abbreviate $\mathbb{Z}_\rho^2 = \ell_2([-\rho,\rho]^2 \cap \mathbb{Z}^2$ and denote by $\ell_2(\mathbb{Z}_\rho^2)$, $\ell_2(T)$, $\ell_2(X)$ the ℓ_2-spaces of sequences indexed by a grid $\mathbb{Z}_\rho^2$, T, and X, respectively. We also need the sampling operator $\mathcal{F}_X : \ell_2(\mathbb{Z}_\rho^2) \to \ell_2(X)$ given by

$$\left(\mathcal{F}_X c_{k,\tilde{k}}\right)(x,y) := \sum_{(k,\tilde{k}) \in \mathbb{Z}_\rho^2} \left(e^{\mathbf{I}kx} e^{\mathbf{J}\tilde{k}y} c_{k,\tilde{k}}\right), \quad (x,y) \in X.$$

By $\mathcal{F}_{TX}$ we denote the restriction of $\mathcal{F}_X$ to sequences with support only on T, thus, an operator acting from $\ell_2(T)$ in $\ell_2(X)$. Also, we have to consider their adjoint operators, $\mathcal{F}_X^* : \ell_2(X) \to \ell_2(\mathbb{Z}_\rho^2)$ and $\mathcal{F}_{TX}^* : \ell_2(X) \to \ell_2(T)$.

The next lemma is the fundamental lemma on which our subsequent investigations will be based.

Lemma 2.2.4. *Let* $c \in \ell_2(\mathbb{Z}_\rho^2)$ *and* $T := \operatorname{supp} c$. *Furthermore, let us assume that* $\mathcal{F}_{TX} : \ell_2(T) \to \ell_2(X)$ *is injective and suppose that there exists a* $P \in \ell_2(\mathbb{Z}_\rho^2)$ *with the following properties:*
(i) $P_{k,\tilde{k}} = \operatorname{sgn}\left(c_{k,\tilde{k}}\right)_{k,\tilde{k}}$ *for all* $(k,\tilde{k}) \in T$,
(ii) $\left|P_{k,\tilde{k}}\right| < 1$ *for all* $(k,\tilde{k}) \notin T$,
(iii) there exists a $\lambda \in \ell_2(X)$ *such that* $P = \mathcal{F}_X^* \lambda$.

Then c *is the unique minimizer to the problem (2.6).*

Proof. Let us assume $X \neq \emptyset$ and $c \neq 0$ to exclude the trivial cases. Furthermore, let us suppose that the vector P exist. Let r be any vector different to c with $\mathcal{F}_X r = \mathcal{F}_X c$. Consider $q := r - c$, then $\mathcal{F}_X q$ vanishes on X. This means that for $r_{k,\tilde{k}}$, $(k,\tilde{k}) \in T$, we have the following estimate

$$\begin{aligned} |r_{k,\tilde{k}}| &= |c_{k,\tilde{k}} + q_{k,\tilde{k}}| = |(c_{k,\tilde{k}} + q_{k,\tilde{k}})\overline{\operatorname{sgn} c_{k,\tilde{k}}} \operatorname{sgn} c_{k,\tilde{k}}| \\ &= |\left(c_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}} + q_{k,\tilde{k}}\,\overline{\operatorname{sgn} c_{k,\tilde{k}}}\right) \operatorname{sgn} c_{k,\tilde{k}}| \end{aligned}$$

$$\begin{aligned}
&= \left| |c_{k,\tilde{k}}| + q_{k,\tilde{k}}\, \overline{\operatorname{sgn} c_{k,\tilde{k}}} \right| |\overline{\operatorname{sgn} c_{k,\tilde{k}}}| \\
&= \left| |c_{k,\tilde{k}}| + q_{k,\tilde{k}}\, \overline{\operatorname{sgn} c_{k,\tilde{k}}} \right| \geq |c_{k,\tilde{k}}| + \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{\operatorname{sgn} c_{k,\tilde{k}}} \right) \\
&= |c_{k,\tilde{k}}| + \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right).
\end{aligned}$$

Thus, for any $(k,\tilde{k}) \in T$ we have $|c_{k,\tilde{k}}| + \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right) \leq |r_{k,\tilde{k}}|$. Otherwise, for $(k,\tilde{k}) \notin T$ we have $\operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right) \leq |q_{k,\tilde{k}}| = |r_{k,\tilde{k}}|$ since $|P_{k,\tilde{k}}| < 1$. Thus

$$||r||_{\ell_1} \geq ||p||_{\ell_1} + \sum_{(k,\tilde{k}) \in [-\rho,\rho]^2 \cap \mathbb{Z}^2} \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right).$$

Now, from condition (iii) we get

$$\sum_{(k,\tilde{k}) \in [-\rho,\rho]^2 \cap \mathbb{Z}^2} \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right) = \operatorname{Re}\left(\sum_{(k,\tilde{k}) \in [-\rho,\rho]^2 \cap \mathbb{Z}^2} q_{k,\tilde{k}}\, \overline{(\mathcal{F}_X^* \lambda)_{k,\tilde{k}}} \right)$$

$$= \operatorname{Re}\left(\sum_{i=1}^{N} \left(\mathcal{F}_X\, q\right)(x_i, y_i) \overline{\lambda(x_i, y_i)} \right) = 0 \tag{2.13}$$

whereas $\mathcal{F}_X\, q$ vanishes. Thus, $\mathcal{F}_X P$ is supported on X and $\mathcal{F}_X q$ vanishes on X. Therefore, $||r||_{\ell_1} \geq ||c||_{\ell_1}$. The equality holds when $||q_{k,\tilde{k}}|| = \operatorname{Re}\left(q_{k,\tilde{k}}\, \overline{P_{k,\tilde{k}}} \right)$ for all $(k,\tilde{k}) \notin T$. Since $||P_{k,\tilde{k}}|| < 1$, this forces q to vanishes outside of T. Taking in account the injectivity of $\mathcal{F}_{TX}$ we have that since $\mathcal{F}_X q$ vanishes on X, q vanishes identically and we have $r = c$. Thus, this shows that c is unique minimizer $c^\sharp$ to the problem (2.6). □

For the invertibility we can state the following obvious lemma.

Lemma 2.2.5. *If $N \geq |T|$ then $\mathcal{F}_{TX}$ is injective almost surely.*

We need to show now that with high probability there exists a P with the required properties so that we can apply Lemma 2.2.4. To show this we proceed as in [69] but with the necessary modifications for the quaternionic case.

We introduce the restriction operator $R_T : \ell_2([-\rho,\rho]^2 \cap \mathbb{Z}^2) \to \ell_2(T)$, $R_T c_{k,\tilde{k}} = c_{k,\tilde{k}}$ for $(k,\tilde{k}) \in T$. Its adjoint $R_T^* = E_T : \ell_2(T) \to \ell_2([-\rho,\rho]^2 \cap \mathbb{Z}^2)$ is the operator that extends a vector outside T by zero, i.e., $(E_T d)_{k,\tilde{k}} = d_{k\tilde{k}}$ for $(k,\tilde{k}) \in T$ and $(E_T d)_{k,\tilde{k}} = 0$ otherwise.

Now, let us assume for the moment that $\mathcal{F}_{TX}^* \mathcal{F}_{TX} : \ell_2(T) \to \ell_2(T)$ is invertible. By Lemma 2.2.5 this is true almost surely if $\tilde{N} \geq |T|$ since $\mathcal{F}_{TX}$ is then injective. In this case we construct P explicitly by

$$P := \mathcal{F}_X^* \mathcal{F}_{TX} (\mathcal{F}_{TX}^* \mathcal{F}_{TX})^{-1} R_T \operatorname{sgn}(c),$$

where as before $T := \operatorname{supp}\left(c_{k,\tilde{k}} \right)_{k,\tilde{k}}$ and $\operatorname{sgn}(c) := \left(\operatorname{sgn} c_{k,\tilde{k}} \right)_{k,\tilde{k}}$. Then clearly P has property

(i) and property (iii) in Lemma 2.2.4 with

$$\lambda := \mathcal{F}_{TX}(\mathcal{F}^*_{TX}\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}\left(c_{k\tilde{k}}\right)_{k\tilde{k}} \in \ell_2(X).$$

We are left with proving that P has property (ii) of Lemma 2.2.4 with high probability.
Let us consider the following matrix operators

$$H := \ell_2(T) \to \ell_2([-\rho,\rho]^2 \cap \mathbb{Z}^2), \quad H_0 := \ell_2(T) \to \ell_2(T),$$

with

$$H := NE_T - \mathcal{F}^*_{TX}\mathcal{F}_{TX} \quad H_0 := NI_T - \mathcal{F}^*_{TX}\mathcal{F}_{TX},$$

where I_T denotes the identity on $l^2(T)$. The matrix H_0 is self-adjoint, and H acts on a vector as

$$(Hc)_{k,\ell} = -\sum_{i=1}^{N} \sum_{\substack{(\tilde{k},\tilde{\ell}) \in T \\ (\tilde{k},\tilde{\ell}) \neq (k,\ell)}} e^{-\mathbf{J}\ell y_i} e^{\mathbf{I}(\tilde{k}-k)x_i} e^{\mathbf{J}\tilde{\ell}y_i} c_{\tilde{k},\tilde{\ell}}, \tag{2.14}$$

By using the matrices H and H_0 we can write

$$P = \left[(NE_T - H)\left(NI_T - H_0\right)^{-1}\right] R_T\mathrm{sgn}(c). \tag{2.15}$$

After joining the two terms into the last representation of P the result (2.15) looks very similar to the one of Rauhut [69], so we just adapt his approach below. Taking into account that we are interested in property (ii) in Lemma 2.2.4 we consider only the values of P on $T^c = ([-\rho,\rho]^2 \cap \mathbb{Z}^2) \setminus T$. Since $R_{T^c}E_T = 0$ we have

$$P_{k,\ell} = \left(-N^{-1}R_{T^c}H\left(I_T - N^{-1}H_0\right)^{-1} R_T\mathrm{sgn}(c)\right)_{k,\ell} \quad \text{for all} \quad (k,\ell) \in T^c. \tag{2.16}$$

Note that in (2.16) both sides are (k,ℓ)-dependent. Let us take a closer look at the inverse quantity $\left(I_T - N^{-1}H_0\right)^{-1}$. We would like to apply the von Neumann series, but that would imply that $\|N^{-1}H_0\| < 1$ which we cannot ensure. But looking at the term

$$\left(I_T - \left(N^{-1}H_0\right)^n\right)^{-1}$$

we get

$$\left(I_T - \left(N^{-1}H_0\right)^n\right)^{-1} = I_T + A_n$$

with

$$A_n := \sum_{r=1}^{\infty} \left(N^{-1}H_0\right)^{rn}. \tag{2.17}$$

provided that $\|(N^{-1}H_0)^n\| < 1$ for some $n \in \mathbb{N}$. Applying the identity formula

$$(1-\gamma)^{-1} = (1-\gamma^n)^{-1}(1+\gamma+\cdots+\gamma^{n-1}) \tag{2.18}$$

we can write

$$\Big(I_T - N^{-1}H_0\Big)^{-1} = (I_T + A_n)\sum_{m=0}^{n-1}\big(N^{-1}H_0\big)^m.$$

Therefore, on the complement of T we have

$$R_{T^c}P_1 = -N^{-1}H(I_T+A_n)\bigg(\sum_{m=0}^{n-1}\big(N^{-1}H_0\big)^m\bigg)R_T\mathrm{sgn}(c) = -\big(P^{(1)}+P^{(2)}\big),$$

where

$$P^{(1)} = S_n\mathrm{sgn}(c) \quad \text{and} \quad P^{(2)} = \frac{1}{N}HA_nR_T(I+S_{n-1})\mathrm{sgn}(c),$$

with $S_n := \sum_{m=0}^{n-1}\big(N^{-1}HR_T\big)^m$.

Our goal is to estimate $\mathbb{P}(\sup_{(k,\ell)\in T^c}|P_{k\ell}| \geq 1)$. For this purpose we consider two numbers $a_1, a_2 > 0$ satisfying $a_1 + a_2 = 1$. Then we have

$$\mathbb{P}\big(\sup_{(k,\ell)\in T^c}|P_{k\ell}| \geq 1\big) \leq \mathbb{P}\big(\big\{\sup_{(k,\ell)\in T^c}|P^{(1)}_{k\ell}| \geq a_1\big\} \cup \big\{\sup_{(k,\ell)\in T^c}|P^{(2)}_{k\ell}| \geq a_2\big\}\big). \tag{2.19}$$

Thus, we can estimate for the first term

$$\begin{aligned}\mathbb{P}\big(|P^{(1)}_{k,\ell}| \geq a_1\big) &= \mathbb{P}\left(\Big|(S_n\mathrm{sgn}(c))_{k,\ell}\Big| \geq a_1\right) \\ &\leq \mathbb{P}\bigg(\sum_{m=1}^{n}|((N^{-1}HR_T)^m\mathrm{sgn}(c))_{k,\ell}| \geq a_1\bigg) =: \mathbb{P}(E_{k,\ell}).\end{aligned} \tag{2.20}$$

Let us attend the $P^{(2)}$ term. Here, it is clear that

$$\begin{aligned}\sup_{(k,\ell)\in T^c}|P^{(2)}_{k,\ell}| &\leq \Big\|P^{(2)}\Big\|_\infty \\ &\leq \big\|N^{-1}HA_n\big\|_{\ell^\infty(T)\to\ell^\infty(\Lambda)}\left(1+\|R_TS_{n-1}\mathrm{sgn}(c)\|_{\ell^\infty(T)}\right),\end{aligned} \tag{2.21}$$

where here $\Lambda := \ell^\infty([-\rho,\rho]^2\cap\mathbb{Z}^2)$ denotes the space of sequences indexed by $[-\rho,\rho]^2\cap\mathbb{Z}^2$ with the supremum norm.

Let us first analyze the term $\|R_TS_{n-1}\mathrm{sgn}(c)\|_{\ell^\infty(T)}$. Here, similarly as in (2.20), we obtain

$$\mathbb{P}\big(|(S_{n-1}\mathrm{sgn}(c))_{k,\ell}| \geq a_1\big) \leq \mathbb{P}\bigg(\sum_{m=1}^{n}|((N^{-1}HR_T)^m\mathrm{sgn}(c))_{k,\ell}| \geq a_1\bigg) = \mathbb{P}(E_{k,\ell})$$

Now, let us analyze the norm of the operator in (2.21). We know that $||A_n||_\infty = \sup_r\sum_s|(A_n)_{r,s}|$. We also know that

$$\big\|N^{-1}HA_n\big\|_\infty \leq \big\|N^{-1}H\big\|_\infty\|A_n\|_{\ell^\infty(T)}. \tag{2.22}$$

with $A_n = \sum_{r=1}^{\infty} \left(N^{-1}H_0\right)^{rn}$.
Let us attend to the term $\|A_n\|_\infty$ first. By Frobenius norm, we have $\|A\|_F^2 := Tr(AA^*) = \sum_{r,s} |A_{rs}|^2$, where $Tr(AA^*)$ denotes the trace of AA^*. Thus, for now, we assume that

$$\left\|\left(N^{-1}H_0\right)^n\right\|_F \leq \kappa < 1. \tag{2.23}$$

From the definition (2.17) of A_n, it follows that

$$\|A_n\|_F = \left\|\sum_{r=1}^{\infty}\left(N^{-1}H_0\right)^{rn}\right\|_F \leq \sum_{r=1}^{\infty}\left\|(N^{-1}H_0)^n\right\|_F^r \leq \sum_{r=1}^{\infty}\kappa^r = \frac{\kappa}{1-\kappa}.$$

Since A_n has $|T|$ columns by Cauchy-Schwarz inequality we get

$$\|A_n\|_\infty^2 \leq \sup_i \sum_j |(A_n)_{i,j}|^2 \leq |T| \, \|A_n\|_F^2 . \tag{2.24}$$

Thus, from (2.23) and $\|S_{n-1}\mathrm{sgn}(c)\|_\infty < a_1$ it follows

$$\sup_{(k,\ell)\in T^c}|P_{k\ell}^{(2)}| \leq (1+a_1)\frac{\kappa}{1-\kappa}\,|T|^{\frac{3}{2}} . \tag{2.25}$$

Therefore, if

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}\,|T|^{-\frac{3}{2}} \tag{2.26}$$

then $\sup_{(k,\ell)\in T^c}|P_{k\ell}^{(2)}| \leq a_2$ as intended.

Also it follows from (2.26) that $\kappa < 1$ and $|T| \geq 1$ (we can exclude the case that if $T = \emptyset$ since $f = 0$ and ℓ_1-minimization will obviously recover f).
Let us remember that in Theorem 2.2.1 $|T|$ is deterministic and in Theorem 2.2.3 $|T|$ is a random variable. We have to treat each case differently. More specifically, we can state the following

1. Consider the second case, where $|T|$ is random by Theorem 2.2.3. If we assume

$$|T| \leq (\alpha+1)\mathbb{E}|T|$$

with $\alpha > 0$ and also assume

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}((\alpha+1)\mathbb{E}|T|)^{-3/2} \tag{2.27}$$

then clearly (2.26) is satisfied and consequently

$$\sup_{t\in T^c}\left|P_{k\ell}^{(2)}\right| \leq a_2.$$

This means that we get from (2.19) the following estimate

$$\mathbb{P}\big(\sup_{(k,\ell)\in T^c}|P_{k,\ell}|\geq 1\big)\leq \mathbb{P}\Big(\bigcup_{(k,\ell)\in T^c}\Big\{\Big|P^{(1)}_{k,\ell}\Big|\geq a_1\Big\}\cup\big\{\,\|R_T\mathrm{sgn}(c)\|_{\ell^\infty(T)}\geq a_1\big\}$$

$$\cup\big\{\,\big\|(N^{-1}H_0)^n\big\|_F\geq\kappa\big\}\cup\big\{\,|T|\geq(\alpha+1)\mathbb{E}\,|T|\Big)$$

$$\leq\mathbb{P}\Big(\bigcup_{(k,\ell)\in[-\rho,\rho]^2\cap\mathbb{Z}^2}E_{k,\ell}\cup\big\{\,\big\|(N^{-1}H_0)^n\big\|_F\geq\kappa\big\}\cup\big\{\,|T|\geq(\alpha+1)\mathbb{E}\,|T|\,\big\}\Big)$$

$$\leq\sum_{(k,\ell)\in[-\rho,\rho]^2\cap\mathbb{Z}^2}\mathbb{P}(E_{k,\ell})+\mathbb{P}\big(\,\big\|(N^{-1}H_0)^n\big\|_F\geq\kappa\big)+\mathbb{P}\big(\,|T|\geq(\alpha+1)\mathbb{E}\,|T|\,\big).\quad(2.28)$$

Note that $|T|$ is the sum of independent random variables. For the third term of (2.28), we obtain

$$\mathbb{P}\big(\,|T|\geq\alpha\mathbb{E}\,|T|+\mathbb{E}\,|T|\,\big)\leq\exp\big(-(\alpha\mathbb{E}\,|T|)^2/(2\mathbb{E}\,|T|+2(\alpha\mathbb{E}\,|T|)/3)\big)$$

$$=e^{\left(-\frac{3\alpha^2}{6+2\alpha}\mathbb{E}|T|\right)}.\quad(2.29)$$

2. For the first case which corresponds to the situation of Theorem 2.2.1, we do not have to treat $|T|$ as a random variable. Using the condition in (2.26), we have

$$\mathbb{P}\big(\sup_{(k,\ell)\in T^c}|P_{k,\ell}|\geq 1\big)\leq\sum_{(k,\ell)\in[-\rho,\rho]^2\cap\mathbb{Z}^2}\mathbb{P}(E_{k,\ell})+\mathbb{P}\big(\,\big\|(N^{-1}H_0)^n\big\|_F\geq\kappa\big).\quad(2.30)$$

Therefore, in the next two sections we will estimate the quantities $\mathbb{P}(E_{k,\ell})$ and $\mathbb{P}\big(\,\big\|(N^{-1}H_0)^n\big\|_F\geq\kappa\big)$, respectively.

2.2.4 Analysis of the powers of H_0

The objective of this section is the calculation of the second term in (2.28) and (2.30). To this end we need to estimate Frobenius norm of our random matrix H_0. For this propose we will estimate $\|H_0^n\|_F^2$ by means of Markov's inequality. We will start with some auxiliary lemmas which will be needed for the proof of Lemma 2.2.8 in which we will determine the expectation value towards the random sampling set $X=\{(x_1,y_1),\ldots,(x_N,y_N)\}$.

Lemma 2.2.6. *Consider $\theta_r\in\mathbb{R}$ and $\underline{a}\in\{-1,1\}$. For*

$$\lambda(a_r,\theta_r)=\begin{cases}\cos\theta_r, & a_r=+1\\ \sin\theta_r, & a_r=-1,\end{cases}$$

it holds

$$2^{n-1}\prod_{r=1}^{n}\lambda(a_r,\theta_r)=\sum_{\alpha_2,\dots,\alpha_n\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_n^{\delta_n}\lambda\Big(\prod_{j=1}^{n}a_j,\theta_1+\alpha_2\theta_2+\cdots+\alpha_n\theta_n\Big)$$

such that $\delta_1:=0$ *and*

$$\delta_n:=\delta\left(\prod_{j=1}^{n-1}a_j,a_n\right) \tag{2.31}$$

such that

$$\delta(a_m,a_{m+1})=\begin{cases}\frac{|a_m|-a_m}{2}, & a_m\cdot a_{m+1}=+1\\ \\ \frac{|a_m|+a_m}{2}, & a_m\cdot a_{m+1}=-1.\end{cases} \tag{2.32}$$

Proof. The proof is given by mathematical induction. In the case of $n=1$ we obtain $\lambda(a_1,\theta_1)=\sum_{\alpha_1\in\{-1,1\}}\lambda(a_1,\theta_1)=\lambda(+1,\theta_1)+\lambda(-1,\theta_1)=\cos\theta_1+\mathbf{I}\sin\theta_1$. Now, let us assume that our equality holds for some natural number n. We have to show that it holds for $n+1$. Here, we can state

$$\begin{aligned}2^{n}\prod_{r=1}^{n+1}\lambda(a_r,\theta_r)&=2^{n-1}\prod_{r=1}^{n}\lambda(a_r,\theta_r)\cdot\lambda(a_{n+1},\theta_{n+1})\\ &=\sum_{\alpha_2,\dots,\alpha_n\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_n^{\delta_n}\lambda\Big(\prod_{j=1}^{n}a_j,\theta_1+\alpha_2\theta_2+\cdots+\alpha_n\theta_n\Big)\cdot\lambda(a_{n+1},\theta_{n+1})\\ &=\sum_{\alpha_2,\dots,\alpha_n,\alpha_{n+1}\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_n^{\delta_n}\cdot\alpha_{n+1}^{\delta_{n+1}}\lambda\Big(\prod_{j=1}^{n+1}a_j,\theta_1+\alpha_2\theta_2+\cdots+\alpha_n\theta_n+\alpha_{n+1}\theta_{n+1}\Big).\end{aligned}$$

□

Lemma 2.2.7. *Let* $n\in\mathbb{N}$. *It holds*

$$\prod_{r=1}^{n}\left(e^{\mathbf{J}A_r}\cos\theta_r+e^{\mathbf{J}C_r}\sin\theta_r\mathbf{I}\right)=\sum_{\underline{a}\in\{-1,1\}^n}e^{\mathbf{J}\sum_{r=1}^{n}\tilde\sigma(a_r)}\mathbf{I}^{\frac{1-\prod_{j=1}^{n}a_j}{2}}\prod_{r=1}^{n}\lambda(a_r,\theta_r), \tag{2.33}$$

where

$$\tilde\sigma(\underline{a},r)=\begin{cases}(-1)^{\pi(r)}A_r, & a_r=+1\\ (-1)^{\pi(r)}C_r, & a_r=-1,\end{cases}\qquad \lambda(a_r,\theta_r)=\begin{cases}\cos\theta_r, & a_r=+1\\ \sin\theta_r, & a_r=-1,\end{cases}$$

such that $\pi(1)=0$ *and* $\pi(n)=\sum_{j=1}^{n-1}\frac{|a_j|-a_j}{2}$ $(n>0)$ *which counts the number of* $\sin$*'s in the vector* $\underline{a}=(\underline{a}_1,\dots,\underline{a}_n)\in\{-1,1\}^n$.

Proof. Again, the proof will be given by mathematical induction.

For the base case $n = 1$ we have

$$e^{\mathbf{J}A_1}\cos\theta_1 + e^{\mathbf{J}C_1}\sin\theta_1\mathbf{I} = \sum_{\underline{a}\in\{-1,1\}} e^{\mathbf{J}\tilde{\sigma}(a_1)}\sigma(a_1) = e^{\mathbf{J}\tilde{\sigma}(+1)}\sigma(+1) + e^{\mathbf{J}\tilde{\sigma}(-1)}\sigma(-1),$$

where $\tilde{\sigma}(+1) = (-1)^{\pi(1)}A_1 = A_1$ and $\tilde{\sigma}(-1) = (-1)^{\pi(1)}C_1 = C_1$. Note that $\pi(1) = 0$ as initial condition. For the induction step, let $n \in \mathbb{N}$ be given and suppose (2.33) is true for n. Then

$$\begin{aligned}
&\prod_{r=1}^{n+1}\left(e^{\mathbf{J}A_r}\cos\theta_r + e^{\mathbf{J}C_r}\sin\theta_r\mathbf{I}\right)\\
&= \prod_{r=1}^{n}\left(e^{\mathbf{J}A_r}\cos\theta_r + e^{\mathbf{J}C_r}\sin\theta_r\mathbf{I}\right)\left(e^{\mathbf{J}A_{n+1}}\cos\theta_{n+1} + e^{\mathbf{J}C_{n+1}}\sin\theta_{n+1}\mathbf{I}\right)\\
&= \prod_{r=1}^{n}\sum_{\underline{a}\in\{-1,1\}^n} e^{\mathbf{J}\sum_{j=1}^{n}\tilde{\sigma}(a_r)}\mathbf{I}^{\frac{1-\prod_{j=1}^{n}a_j}{2}}\prod_{r=1}^{n}\lambda(a_r,\theta_r)\left(e^{\mathbf{J}A_{n+1}}\cos\theta_{n+1} + e^{\mathbf{J}C_{n+1}}\sin\theta_{n+1}\mathbf{I}\right)\\
&= \prod_{r=1}^{n}\sum_{\underline{a}\in\{-1,1\}^n} e^{\mathbf{J}\sum_{r=1}^{n}\tilde{\sigma}(a_r)+(-1)^{\pi(n+1)}A_{n+1}}\mathbf{I}^{\frac{1-\prod_{j=1}^{n}a_j}{2}}\prod_{r=1}^{n}\lambda(a_r,\theta_r)\cos\theta_{n+1}\\
&+ \prod_{r=1}^{n}\sum_{\underline{a}\in\{-1,1\}^n} e^{\mathbf{J}\sum_{r=1}^{n}\tilde{\sigma}(a_r)+(-1)^{\pi(n+1)}C_{n+1}}\mathbf{I}^{\frac{1-\prod_{j=1}^{n}a_j}{2}}\prod_{r=1}^{n}\lambda(a_r,\theta_r)\sin\theta_{n+1}\mathbf{I}\\
&= \prod_{r=1}^{n+1}\sum_{\underline{\underline{a}}\in\{-1,1\}^{n+1}} e^{\mathbf{J}\sum_{r=1}^{n+1}\tilde{\sigma}(a_r)}\mathbf{I}^{\frac{1-\prod_{j=1}^{n+1}a_j}{2}}\prod_{j=1}^{n+1}\lambda(a_r,\theta_r),
\end{aligned}$$

such that $\underline{\underline{a}} = (\underline{a}, a_{n+1}) \in \{-1,1\}^{n+1}$ and $a_{n+1} = \pm 1$. □

Using the above lemmas we can now estimate the expectation value of the Frobenius norm of our random matrix H_0^n. The need of these lemmas stems from the fact that we cannot use a direct approach via exponentials due the non-commutativity of quaternions.

Lemma 2.2.8. *It holds*

$$\mathbb{E}_X\left[\|H_0^n\|_F^2\right] = \sum_{t=1}^{\min\{n,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2n,t)}\sum_{\substack{(k_1,\ell_1),\dots,(k_{2n},\ell_{2n})\in T\\(k_j,\ell_j)\neq(k_{j+1},\ell_{j+1})}}$$

$$\times\prod_{A\subset\mathcal{A}}\sum_{\substack{\underline{a}\in\{-1,1\}^r\\a_1\cdots a_r=+1}}\delta\left(\sum_{r\in A}\tilde{\varrho}(a_r)\right)2^{1-|A|}\sum_{\alpha_2,\dots,\alpha_n\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_{|A|}^{\delta_{|A|}}\delta\left(\theta_1+\sum_{r\in A}\alpha_s\theta_s\right),$$

where $\delta(n)$ denotes the Kronecker δ_{0n} and $(k_{2n+1},\ell_{2n+1}) = (k_1,\ell_1)$ and

$$\tilde{\varrho}(\underline{a},\ell_r) = \begin{cases}(-1)^{\pi(r)}(\ell_{r+1}-\ell_r), & a_r = +1\\ -(-1)^{\pi(r)}(\ell_{r+1}+\ell_r), & a_r = -1,\end{cases}\qquad \theta_s = (k_{s+1}-k_s)$$

such that $\pi(|A|) = \sum_{j=1}^{|A|-1} \frac{|a_j|-a_j}{2}$.

Proof. Since H_0^n is self adjoint we have from the definition $\mathbb{E}_X\left[\|H_0^n\|_F^2\right] = \mathbb{E}_X\left[\mathrm{tr}H_0^{2n}\right]$. Consider

$$H_0\left[(k_1,\ell_1),(k_2,\ell_2)\right] = (1-\delta_{k_1k_2}\delta_{\ell_1\ell_2})\sum_{i_1=1}^{N}\left[e^{-\mathbf{J}\ell_1 y_{i_1}}e^{\mathbf{I}(k_2-k_1)x_{i_1}}e^{\mathbf{J}\ell_2 y_{i_1}}\right],\quad (k_1,\ell_1),(k_2,\ell_2)\in T.$$

Algebraic operations with quaternions constitute a big handicap because quaternions do not commute. Thus we have to find a new strategy. What plays to our advantage are the anti-commutativity of the basis elements, that is, $\mathbf{IJ} = -\mathbf{JI}$. Moreover, this means that $\mathbf{I}e^{\mathbf{J}y} = e^{-\mathbf{J}y}\mathbf{I}$. Thus, we can simplify our expression as

$$e^{-\mathbf{J}\ell_r y_{i_r}}e^{\mathbf{I}(k_{r+1}-k_r)x_{i_r}}e^{\mathbf{J}\ell_{r+1}y_{i_r}}$$

$$= e^{\mathbf{J}(\ell_{r+1}-\ell_r)y_{i_r}}\cos\left([k_{r+1}-k_r]x_{i_r}\right) + e^{-\mathbf{J}(\ell_{r+1}+\ell_r)y_{i_r}}\sin\left([k_{r+1}-k_r]x_{i_r}\right)\mathbf{I}. \tag{2.34}$$

Then we write the H_0^2 as

$$\begin{aligned}
&H_0^2\left[(k_1,\ell_1),(k_3,\ell_3)\right]\\
&= \sum_{\substack{(k_2,\ell_2)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,2.}} H_0\left[(k_1,\ell_1),(k_2,\ell_2)\right]H_0\left[(k_2,\ell_2),(k_3,\ell_3)\right]\\
&= \sum_{\substack{i_1=1\\ i_2=1}}^{N}\ \sum_{\substack{(k_2,\ell_2)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,2.}} \left[e^{-\mathbf{J}\ell_1 y_{i_1}}e^{\mathbf{I}(k_2-k_1)x_{i_1}}e^{\mathbf{J}\ell_2 y_{i_1}}\right]\\
&\qquad\times\left[e^{-\mathbf{J}\ell_2 y_{i_2}}e^{\mathbf{I}(k_3-k_2)x_{i_2}}e^{\mathbf{J}\ell_3 y_{i_2}}\right].
\end{aligned}$$

For H_0^4 we get

$$\begin{aligned}
&H_0^4\left[(k_1,\ell_1),(k_5,\ell_5)\right]\\
&= \sum_{\substack{(k_2,\ell_2),(k_4,\ell_4)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,2,3,4}} H_0\left[(k_1,\ell_1),(k_2,\ell_2)\right]H_0\left[(k_2,\ell_2),(k_3,\ell_3)\right]\\
&\qquad\times H_0\left[(k_3,\ell_3),(k_4,\ell_4)\right]H_0\left[(k_4,\ell_4),(k_5,\ell_5)\right]\\
&= \sum_{\substack{i_1=1\\ \vdots\\ i_4=1}}^{N}\ \sum_{\substack{(k_2,\ell_2),(k_4,\ell_4)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,2,3,4}} \left[e^{-\mathbf{J}\ell_1 y_{i_1}}e^{\mathbf{I}(k_2-k_1)x_{i_1}}e^{\mathbf{J}\ell_2 y_{i_1}}\right]
\end{aligned}$$

$$\times \left[e^{-\mathbf{J}\ell_2 y_{i_2}} e^{\mathbf{I}(k_3-k_2)x_{i_2}} e^{\mathbf{J}\ell_3 y_{i_2}}\right] \left[e^{-\mathbf{J}\ell_3 y_{i_3}} e^{\mathbf{I}(k_4-k_3)x_{i_3}} e^{\mathbf{J}\ell_4 y_{i_3}}\right] \left[e^{-\mathbf{J}\ell_4 y_{i_4}} e^{\mathbf{I}(k_5-k_4)x_{i_4}} e^{\mathbf{J}\ell_5 y_{i_4}}\right].$$

For the $2n$-th power of the matrix H_0, we write

$$\begin{aligned} & H_0^{2n}\left[(k_1,\ell_1),(k_{2n+1},\ell_{k_{2n+1}})\right] \\ = & \sum_{\substack{(k_2,\ell_2),\dots,(k_{2n},\ell_{2n})\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1})}} H_0\left[(k_1,\ell_1),(k_2,\ell_2)\right]\cdots H_0\left[(k_{2n},\ell_{2n})\,(k_{2n+1},\ell_{k_{2n+1}})\right] \\ = & \sum_{\substack{i_1=1 \\ \vdots \\ i_{2n}=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_{2n},\ell_{2n})\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\dots,2n}} \left[\prod_{r=1}^{2n} e^{-\mathbf{J}\ell_r y_{i_r}} e^{\mathbf{I}(k_{r+1}-k_r)x_{i_r}} e^{\mathbf{J}\ell_{r+1} y_{i_r}}\right]. \end{aligned}$$

Since we need the diagonal element of the $2n$-th power of our matrix to calculate the norm $\|H_0^n\|_F^2 = \mathrm{Tr} H_0^{2n}$ we require $(k_{2n+1},\ell_{k_{2n+1}}) = (k_1,\ell_1)$. Therefore, using the linearity of the expectation value it follows

$$\begin{aligned} & \mathbb{E}_X\left[\mathrm{Tr} H_0^{2n}\right] \\ = & \sum_{\substack{i_1=1 \\ \vdots \\ i_{2n}=1}}^{N} \sum_{\substack{(k_1,\ell_1),\dots,(k_{2n},\ell_{2n})\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1})}} \mathbb{E}_X\left[\prod_{r=1}^{2n} e^{-\mathbf{J}\ell_r y_{i_r}} e^{\mathbf{I}(k_{r+1}-k_r)x_{i_r}} e^{\mathbf{J}\ell_{r+1} y_{i_r}}\right]. \end{aligned}$$

Furthermore, some indices i_r might be the same which means that we cannot use directly the product rule for the expectation value since it is only valid for independent random variables. This is where we consider a set partition. We associate a partition $\mathcal{A} = (A_1, A_2, \dots, A_t)$ of $\{1,\dots,2n\}$ to a certain vector $i_1,\dots,i_{2n}$ such that $i_r = i_{r'}$ if and only if r and r' are contained in the same set $A_i \in \mathcal{A}$. This is allows us to unambiguously write i_A instead of i_r if $r \in A$. Now, the independence of the variables (x_A, y_A) yields

$$\begin{aligned} & \mathbb{E}_X\left[\prod_{r=1}^{2n} \exp\left(-\mathbf{J}\ell_r y_{i_r}\right) \exp\left(\mathbf{I}(k_{r+1}-k_r)x_{i_r}\right) \exp\left(\mathbf{J}\ell_{r+1} y_{i_r}\right)\right] \\ = & \mathbb{E}_X\left[\prod_{A\in\mathcal{A}}\prod_{r\in A} \exp\left(-\mathbf{J}\ell_r y_A\right) \exp\left(\mathbf{I}(k_{r+1}-k_r)x_A\right) \exp\left(\mathbf{J}\ell_{r+1} y_A\right)\right] \\ = & \prod_{A\in\mathcal{A}} \mathbb{E}_X\left[\prod_{r\in A} \exp\left(-\mathbf{J}\ell_r y_A\right) \exp\left(\mathbf{I}(k_{r+1}-k_r)x_A\right) \exp\left(\mathbf{J}\ell_{r+1} y_A\right)\right]. \end{aligned} \tag{2.35}$$

Let us introduce now

$$\tilde{\sigma}(\underline{a}, \ell_r) = \begin{cases} (-1)^{\pi(r)}(\ell_{r+1} - \ell_r)y_A, & a_r = +1 \\ (-1)^{\pi(r)}(-\ell_{r+1} - \ell_r)y_A, & a_r = -1, \end{cases}$$

and

$$\lambda(a_r, \theta_r) = \begin{cases} \cos(k_{r+1} - k_r)x_A, & a_r = +1 \\ \sin(k_{r+1} - k_r)x_A, & a_r = -1 \end{cases},$$

and by applying Lemma 2.2.7 we get for the expectation value in (2.35)

$$\mathbb{E}_X \left(\sum_{\underline{a} \in \{-1,1\}^r} e^{\mathbf{J} \sum_{r \in A} \tilde{\sigma}(a_r)} \prod_{r \in A} \lambda(a_r, \theta_r) \right)$$

$$= \mathbb{E}_X \left(\sum_{\substack{\underline{a} \in \{-1,1\}^r \\ a_1 \cdots a_r = +1}} e^{\mathbf{J} \sum_{r \in A} \tilde{\sigma}(a_r)} \prod_{r \in A} \lambda(a_r, \theta_r)) + \sum_{\substack{\underline{a} \in \{-1,1\}^r \\ a_1 \cdots a_r = -1}} e^{\mathbf{J} \sum_{r \in A} \tilde{\sigma}(a_r)} \mathbf{I} \prod_{r \in A} \lambda(a_r, \theta_r) \right)$$

$$\begin{aligned} = \quad & \mathbb{E}_X \left(\sum_{\substack{\underline{a} \in \{-1,1\}^r \\ a_1 \cdots a_r = +1}} e^{\mathbf{J} \sum_{r \in A} \tilde{\sigma}(a_r)} \prod_{r \in A} \lambda(a_r, \theta_r) \right) \\ & + \quad \mathbb{E}_X \left(\sum_{\substack{\underline{a} \in \{-1,1\}^r \\ a_1 \cdots a_r = -1}} e^{\mathbf{J} \sum_{r \in A} \tilde{\sigma}(a_r)} \mathbf{I} \prod_{r \in A} \lambda(a_r, \theta_r) \right). \end{aligned} \tag{2.36}$$

The last line in (2.36) can be analyzed as follows: when $a_1 \cdots a_r = +1$ then the term is independent of $\mathbf{I}$ and when $a_1 \cdots a_r = -1$ the term depends on $\mathbf{I}$. This can be easily seen as in the first case the number of $a_j = -1$ which corresponds to the terms containing $\mathbf{I}$ is even. For this reason the element $\mathbf{I}$ vanishes because $\mathbf{I}^{2r} = \pm 1$, with $r \in \mathbb{N}$. For the second term in (2.36) the number of terms with $a_j = -1$ is odd, matching with the number of sin-functions in the product. That is means the element $\mathbf{I}$ never vanishes for the second term since $\mathbf{I}^{2r-1} = \pm \mathbf{I}$, with $r \in \mathbb{N}$.
We emphasized here that the expectation value of the second term vanishes because it will be given in terms of sin-functions which vanish when x_{s_A} is integrated over $[0, 2\pi]$. Therefore, only the first

term is surviving. In turn, the first term has 2^{n-1} sub-terms. Thus, we can state

$$\mathbb{E}_X\left[\sum_{\substack{\underline{a}\in\{-1,1\}^r\\ a_1\cdots a_r=+1}} e^{\mathbf{J}\sum_{r\in A}\tilde{\sigma}(a_r)}\prod_{r\in A}\lambda(a_r,\theta_r)\right]$$
$$=\sum_{\substack{\underline{a}\in\{-1,1\}^r\\ a_1\cdots a_r=-1}}\int_{[0,2\pi]} e^{\mathbf{J}\sum_{r\in A}\tilde{\sigma}(a_r)}dy\int_{[0,2\pi]}\prod_{r\in A}\lambda(a_r,\theta_r)dx$$

where

$$\tilde{\sigma}(a_r)=\begin{cases}(-1)^{\pi(r)}(\ell_{r+1}-\ell_r)y_r, & a_r=+1\\ -(-1)^{\pi(r)}(\ell_{r+1}+\ell_r)y_r, & a_r=-1,\end{cases}$$

and

$$\lambda(a_r,\theta_r)=\begin{cases}\cos[(k_{r+1}-k_r)x_r], & a_r=+1\\ \sin[(k_{r+1}-k_r)x_r], & a_r=-1,\end{cases}$$

such that $\pi(n)=\sum_{j=1}^{n-1}\frac{1-a_j}{2}$. The last quantity $\pi(n)$ does nothing else then counting the number of $\sin$-functions. By invoking Lemma 2.2.6 for the product altogether we obtain

$$\mathbb{E}_X\left[\exp\left(\mathbf{J}\sum_{r\in A}\tilde{\sigma}(\underline{a},\ell_r)\right)\prod_{s\in A}\lambda(a_r,\theta_r)\right]$$
$$=\sum_{\substack{\underline{a}\in\{-1,1\}^r\\ a_1\cdots a_r=+1}}\delta\left(\sum_{r\in A}\tilde{\varrho}(a_r)\right)2^{1-|A|}\sum_{\alpha_2,\ldots,\alpha_n\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_{|A|}^{\delta_{|A|}}\delta\left(\theta_1+\sum_{r\in A}\alpha_r\theta_r\right),$$

such that

$$\tilde{\varrho}(\underline{a},\ell_r)=\begin{cases}(-1)^{\pi(r)}(\ell_{r+1}-\ell_r), & a_r=+1\\ -(-1)^{\pi(r)}(\ell_{r+1}+\ell_r), & a_r=-1.\end{cases}\tag{2.37}$$

Furthermore, the condition $|A|\geq 2$ for all $A\in\mathcal{A}$ should be satisfied, i.e., we consider partitions $P(2n,t)$ with $t\geq 2$. Moreover, the number of vectors $(A_1,A_2,\ldots,A_t)\in\{1,\ldots,N\}^t$ with different entries is exactly $N\cdots(N-t+1)=N!/(N-t)!$ if $N\geq t$ and 0 if $N\leq t$. □

For later reference let us introduce the following notation

$$C_{\mathbb{H}}(\mathcal{A},T) := \sum_{\substack{(k_1,\ell_1),\ldots,(k_{2n},\ell_{2n})\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1})}} \sum_{\substack{\underline{a}\in\{-1,1\}^r\\ a_1\cdots a_r=+1}} \prod_{A\in\mathcal{A}} \tag{2.38}$$

$$\delta\left(\sum_{s\in A}\tilde{\varrho}(a_s)\right) 2^{1-|A|} \sum_{\alpha_2,\ldots,\alpha_n\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_{|A|}^{\delta_{|A|}}\delta\left(\theta_1+\sum_{r\in A}\alpha_r\theta_r\right),$$

such that

$$\tilde{\varrho}(\underline{a},\ell_s) = \begin{cases}\beta_s(\ell_{s+1}-\ell_s), & a_s=+1\\ -\beta_s(\ell_{s+1}+\ell_s), & a_s=-1.\end{cases}$$

with $\beta_s = (-1)^{\pi(s)}$ where $\pi(s)=\sum_{j=1}^{n-1}\frac{|a_j|-a_j}{2}$.

2.2.5 Analysis of $\mathbb{P}(E_{k,\ell})$

While we have the estimate for the Frobenius norm of H_0^n in the previous section we still need to study the probability $\mathbb{P}(E_{k,\ell})$, i.e. the first term in (2.28) and (2.30). Similar to [69] consider the numbers $\beta_m > 0$, $m = 1,\ldots,n$, such that

$$\sum_{m=1}^{n}\beta_m = a_1$$

and $K_m \in \mathbb{N}$, $m = 1,\ldots,n$, some natural numbers. Let $(k,\ell)\in[-\rho,\rho]^2\cap\mathbb{Z}^2$. By applying Markov's inequality it follows

$$\begin{aligned}
\mathbb{P}(E_{k,\ell}) &= \mathbb{P}\left(\sum_{m=1}^{n}|((N^{-1}HR_T)^m\mathrm{sgn}(c))_{k,\ell}|\geq a_1\right)\\
&\leq \sum_{m=1}^{n}\mathbb{P}\left(N^{-m}|((HR_T)^m\mathrm{sgn}(c))_{k,\ell}|\geq\beta_m\right)\\
&= \sum_{m=1}^{n}\mathbb{P}\left(N^{-2mK_m}|((HR_T)^m\mathrm{sgn}(c))_{k,\ell}|^{2K_m}\geq\beta_m^{2K_m}\right)\\
&\leq \sum_{m=1}^{n}\mathbb{E}_X\left(|((HR_T)^m\mathrm{sgn}(c))_{k,\ell}|^{2K_m}\right)N^{-2mK_m}\beta_m^{-2K_m}.
\end{aligned} \tag{2.39}$$

Let us consider $\beta_m = \beta^{n/K_m}$, i.e., $\beta_m^{-2K_m}=\beta_m^{-2n}$. Now, from (2.39) it follows

$$\mathbb{P}(E_{k,\ell}) \leq \beta_m^{-2n}\sum_{m=1}^{n}\mathbb{E}_X\left(|((HR_T)^m\mathrm{sgn}(c))_{k,\ell}|^{2K_m}\right)N^{-2mK_m} \tag{2.40}$$

while the condition $a_1 = \sum_{m=1}^{n} \beta_m$ can be written as

$$a_1 = a = \sum_{m=1}^{n} \beta^{n/K_m} < 1.$$

The following lemma attends the expectation value appearing in (2.40). Lemma 2.2.9 is similar to Lemma 2.2.8, it requires much more work and, unfortunately, lengthy calculations. That is why we choose to put the lengthy calculations of the proof in Appendix B.

Lemma 2.2.9. *For a sequence $c = (c_{k,\ell}) \in \ell_2([-\rho,\rho]^2 \cap \mathbb{Z}^2)$ with $\operatorname{supp} c = T$, it holds*

$$\mathbb{E}_X \left[\left| \left((HR_T)^m \operatorname{sgn}(c) \right)_{k\ell} \right|^{2K} \right]$$

$$\leq \sum_{t=1}^{\min\{Km,N\}} \frac{N!}{(N-t)!} \sum_{\mathcal{A} \in P(2Km,t)} \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)}) \in T \\ \vdots \\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)}) \in T \\ (k_j^{(p)},\ell_j^{(p)}) \neq (k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{3-m}$$

$$\sum_{(r_1,\ldots,r_K) \subset V(\{1,\ldots,16\},K)} \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = \pm 1 \\ \vdots \\ a_1^{(K)} \cdots a_m^{(K)} = \pm 1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \vdots \\ \alpha_2^{(K)},\ldots,\alpha_m^{(K)} \in \{-1,1\}}}$$

$$\prod_{A \in \mathcal{A}} \delta \left(\sum_{(r,p) \in A} \alpha_r^{(p)} \theta_r^{(p)} \right) \delta \left(\sum_{(s,p) \in A} \beta_s^{(p)} \phi_s^{(p)} \right),$$

where $\theta_r^{(p)} = \left(k_{r+1}^{(p)} - k_r^{(p)} \right)$ *and* $\phi_s^{(p)} = \begin{cases} (\ell_{s+1}^{(p)} - \ell_s^{(p)}), & a_s = +1 \\ -(\ell_{s+1}^{(p)} + \ell_s^{(p)}), & a_s = -1, \end{cases}$ *which* $\alpha_r^{(p)} \in \{-1,1\}$ *and* $\beta_s^{(p)} = (-1)^{\pi(s)}$ *such that* $\pi(s) = \sum_{j=1}^{s-1} \frac{|a_j| - a_j}{2}$ *which counts the number of* $\sin$*'s.*

Proof. See Appendix B. We want to compute the expectation value $\mathbb{E}_X$ of the Kth-power of the modulus, i.e., the expectation value of the quantity $|((HR_T)^m \operatorname{sgn}(c_{k\ell}))_{k\ell}|^{2K}$. But there exist a major problem due the non-commutativity in the quaternionic setting. While in the previous proof we could still use the anti-commutativity of the basis elements **I** and **J** we now have to take into account the quaternion-valued coefficients. That means we have to compute first $(HR_T)^m$ and then multiply it by the coefficient vector signal. That we have to do in the way that we write the previous quantity as $\left(|\left((HR_T)^m \sigma \right)_{k\ell}|^2 \right)^K = \left(Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 \right)^K$. Of course, breaking the power of a quaternion in several small pieces and joining afterwards it is a hard task. We leave the the calculation of the huge formulas for Appendix B. Thus, we get

$$\left((HR_T) \left[(k_1,\ell_1), (k_{m+1},\ell_{m+1}) \right] \right)^m =$$

$$= \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} HR_T\left[(k_1,\ell_1),(k_2,\ell_2)\right]\cdots HR_T\left[(k_m,\ell_m)\,(k_{m+1},\ell_{m+1})\right]$$

$$= \sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \left[\prod_{r=1}^{m} e^{-\mathbf{J}\ell_r y_{i_r}} e^{\mathbf{I}(k_{r+1}-k_r)x_{i_r}} e^{\mathbf{J}\ell_{r+1}y_{i_r}}\right]$$

$$= \sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\underline{a}\in\{-1,1\}^m} e^{\mathbf{J}\sum_{r=1}^{m}\beta_r\phi_r} \prod_{s=1}^{m}\lambda(a_s,\theta_s)$$

$$= \sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^{m}\beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\lambda\left(\prod_{j=1}^{m}a_j,\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^{m}\beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\lambda\left(\prod_{j=1}^{m}a_j,\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right).$$

By according the our previous Lemma 2.2.6, we have

$$\left((HR_T)\left[(k_1,\ell_1),(k_{m+1},\ell_{m+1})\right]\right)^m$$

$$= \sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^{m}\beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m} \sin\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)\mathbf{I}$$

$$= \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r\right)+\mathbf{J}\sin\left(\sum_{r=1}^m \beta_r\phi_r\right)\right) \sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m} \cos\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r\right)+\mathbf{J}\sin\left(\sum_{r=1}^m \beta_r\phi_r\right)\right) \sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_n^{\delta_m} \sin\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)\mathbf{I}$$

$$= \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^m \beta_r\phi_r\right) \sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m} \cos\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^m \beta_r\phi_r\right) \sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m} \cos\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)\mathbf{J}$$

$$+ \sum_{\substack{i_1=1 \\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1\cdots a_m=-1}} 2^{1-m}$$
$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right) \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{I}$$
$$+ \sum_{\substack{i_1=1 \\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\, j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1\cdots a_m=-1}} 2^{1-m}$$
$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right) \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{JI}$$
$$= P_0+\mathbf{I}P_1+\mathbf{J}P_2-\mathbf{K}P_3.$$

Let us consider $\sigma_m=\sigma_m(k_1,\ell_1)=\sigma_{0m}(k_1,\ell_1)+\mathbf{I}\sigma_{1m}(k_1,\ell_1)+\mathbf{J}\sigma_{2m}(k_1,\ell_1)+\mathbf{K}\sigma_{3m}(k_1,\ell_1)$. We are interested to compute

$$\begin{aligned}
&\sum_{(k_1,\ell_1)\in T}[P_0+\mathbf{I}P_1+\mathbf{J}P_2-\mathbf{K}P_3] \\
&\quad\times[\sigma_{0m}(k_1,\ell_1)+\mathbf{I}\sigma_{1m}(k_1,\ell_1)+\mathbf{J}\sigma_{2m}(k_1,\ell_1)+\mathbf{K}\sigma_{3m}(k_1,\ell_1)] \\
&= \sum_{(k_1,\ell_1)\in T}\Big[(P_0\sigma_{0m}(k_1,\ell_1)-P_1\sigma_{1m}(k_1,\ell_1)-P_2\sigma_{2m}(k_1,\ell_1)+P_3\sigma_{3m}(k_1,\ell_1)) \\
&\quad+(P_0\sigma_{1m}(k_1,\ell_1)+P_1\sigma_{0m}(k_1,\ell_1)+P_2\sigma_{\rho_0 3}+P_3\sigma_{2m}(k_1,\ell_1))\,\mathbf{I} \\
&\quad+(P_0\sigma_{2m}(k_1,\ell_1)-P_1\sigma_{3m}(k_1,\ell_1)+P_2\sigma_{0m}(k_1,\ell_1)-P_3\sigma_{1m}(k_1,\ell_1))\,\mathbf{J} \\
&\quad+(P_0\sigma_{03m}(k_1,\ell_1)+P_1\sigma_{2m}(k_1,\ell_1)-P_2\sigma_{1m}(k_1,\ell_1)-P_3\sigma_{0m}(k_1,\ell_1))\,\mathbf{K}\Big] \\
&= \sum_{(k_1,\ell_1)\in T}(P_0\sigma_{0m}(k_1,\ell_1)-P_1\sigma_{1m}(k_1,\ell_1)-P_2\sigma_{2m}(k_1,\ell_1)+P_3\sigma_{3m}(k_1,\ell_1)) \\
&\quad+\sum_{(k_1,\ell_1)\in T}(P_0\sigma_{1m}(k_1,\ell_1)+P_1\sigma_{0m}(k_1,\ell_1)+P_2\sigma_{\rho_0 3}+P_3\sigma_{2m}(k_1,\ell_1))\,\mathbf{I} \\
&\quad+\sum_{(k_1,\ell_1)\in T}(P_0\sigma_{2m}(k_1,\ell_1)-P_1\sigma_{3m}(k_1,\ell_1)+P_2\sigma_{0m}(k_1,\ell_1)-P_3\sigma_{1m}(k_1,\ell_1))\,\mathbf{J} \\
&\quad+\sum_{(k_1,\ell_1)\in T}(P_0\sigma_{03m}(k_1,\ell_1)+P_1\sigma_{2m}(k_1,\ell_1)-P_2\sigma_{1m}(k_1,\ell_1)-P_3\sigma_{0m}(k_1,\ell_1))\,\mathbf{K} \\
&= Q_0+Q_1\mathbf{I}+Q_2\mathbf{J}+Q_3\mathbf{K}.
\end{aligned}$$

Let us consider the expectation value applied to the sum. As in the proof of Lemma 2.2.8 we have to take into account that some of the indices $i_r^{(p)}$ might coincide. This means that we have to use again set partitions. Let $(i_r^{(p)})_{r=1,\ldots,m}^{p=1,\ldots,2K} \subset \{1,\ldots,N\}^{2Km}$ be a vector of indices and let $\mathcal{A}=(A_1,A_2,\ldots,A_t)\,,\ A_i\subset\{1,\ldots,m\}\times\{1,\ldots,2K\}$ be a corresponding partition such that (r,p) and (r',p') are contained in the same block if and only if $i_r^{(p)}=i_{r'}^{(p')}$. For some $A\in\mathcal{A}$ we may unambiguously write i_A instead of $i_r^{(p)}$ if $(r,p)\in A$. Once again, if $A\subset\mathcal{A}$ contains only one element then the last expression vanishes due to the condition $(k_r^{(p)},\ell_r^{(p)})\neq(k_{r-1}^{(p)},\ell_{r-1}^{(p)})$. Thus, we only need to consider partitions $\mathcal{A}\in P(2Km,t)$ with $t>1$. Now we are able to rewrite the inequality as

$$\mathbb{E}_X\left[|((HR_T)^m\operatorname{sgn}(c_{k\ell}))_{k\ell}|^{2K}\right]$$
$$\leq \sum_{t=1}^{\min\{Km,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2Km,t)}\sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots\\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{3-m}$$
$$\sum_{(r_1,\ldots,r_K)\subset V(\{1,\ldots,16\},K)}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=\pm1\\ \vdots\\ a_1^{(K)}\cdots a_m^{(K)}=\pm1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \vdots\\ \alpha_2^{(K)},\ldots,\alpha_m^{(K)}\in\{-1,1\}}}\ \prod_{A\in\mathcal{A}}$$
$$\prod_{A\in\mathcal{A}}\delta\left(\sum_{(r,p)\in A}\alpha_r^{(p)}\theta_r^{(p)}\right)\delta\left(\sum_{(s,p)\in A}\beta_s^{(p)}\phi_s^{(p)}\right),$$

where $\theta_r^{(p)}=\left(k_{r+1}^{(p)}-k_r^{(p)}\right)$ and $\phi_s^{(p)}=\begin{cases}(\ell_{s+1}^{(p)}-\ell_s^{(p)}), & a_s=+1\\ -(\ell_{s+1}^{(p)}+\ell_s^{(p)}), & a_s=-1,\end{cases}$ which $\alpha_r^{(p)}\in\{-1,1\}$ and $\beta_s^{(p)}=(-1)^{\pi(s)}$ with $\pi(s)=\sum_{j=1}^{s-1}\frac{|a_j|-a_j}{2}$. □

To simplify our work in the next subsection we abbreviate

$$B_{\mathbb{H}}(\mathcal{A},T)=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots\\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{3-m} \tag{2.41}$$

$$\sum_{(r_1,\dots,r_K)\subset V(\{1,\dots,16\},K)} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=\pm 1 \\ \vdots \\ a_1^{(K)}\cdots a_m^{(K)}=\pm 1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \vdots \\ \alpha_2^{(K)},\dots,\alpha_m^{(K)}\in\{-1,1\}}}$$

$$\prod_{A\in\mathcal{A}} \delta\left(\sum_{(r,p)\in A} \alpha_r^{(p)}\theta_r^{(p)}\right)\delta\left(\sum_{(s,p)\in A}\beta_s^{(p)}\phi_s^{(p)}\right).$$

2.2.6 Proof of Theorem 2.2.1

Using the lemmas from the previous subsection, in particular Lemma 2.2.8, we will give now the proof of Theorem 2.2.1. The absolute value of the quantity $C_{\mathbb{H}}(\mathcal{A},T)$ as defined in (2.38) depends on T and $\mathcal{A}\in P(2n,t)$. Here the indices $((k_1,\ell_1),\dots,(k_{2n},\ell_{2n}))\in T^{2n}$ are subjected to the $|\mathcal{A}|=t$ linear constraints $\sum_{r\in A}(k_{r+1}-k_r)=0$ and $\sum_{s\in A}(\ell_{s+1}\pm\ell_s)=0$ for all $A\in\mathcal{A}$. These constraints are independent except for $\sum_{r=1}^{2n}(k_{r+1}-k_r)=0$ and $\sum_{s=1}^{2n}(\ell_{s+1}\pm\ell_s)=0$. Thus, from (2.38) we can estimate

$$C_{\mathbb{H}}(\mathcal{A},T)\le |T|^{2n-t+1}\le M^{2n-t+1}. \tag{2.42}$$

By Lemma 2.2.8 we obtain (since in Theorem 2.2.1 T is not random which means $\mathbb{E}=\mathbb{E}_X$)

$$\mathbb{E}\left[\|H_0^n\|_F^2\right]\le \sum_{t=1}^{\min\{n,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2n,t)}|T|^{2n-t+1}\le M^{2n+1}\sum_{t=1}^{n}\left(\frac{N}{M}\right)^t S_2(2n,t),$$

where $S_2(n,t)=|P(2n,t)|$ are the associated Stirling numbers of the second kind. Let us put $\theta=\frac{N}{M}$. By Markov's inequality it follows

$$\begin{aligned}\mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\|_F\ge\kappa\right) &= \mathbb{P}\left(\|H_0^n\|_F^2\ge N^{2n}\kappa^2\right)\le N^{-2n}\kappa^{-2}\,\mathbb{E}\left[\|H_0^n\|_F^2\right]\\ &\le \kappa^{-2}\,M\,\theta^{-2n}F_{2n}(\theta)=\kappa^{-2}\;M\,G_{2n}(\theta).\end{aligned}$$

Let us note that $\kappa<1$. This implies $\left\|\left(N^{-1}H_0\right)^n\right\|_F\le\kappa$ and, therefore, $\left(I_T-\left(\tilde{N}^{-1}H_0\right)^n\right)$ is invertible by identity (2.18) and the inverse is given by the von-Neumann series and also $[\mathcal{F}_{TX}^*\mathcal{F}_{TX}]=N\left(I_T-N^{-1}H_0\right)$ is invertible. In particular, $\mathcal{F}_{TX}$ is injective.

Let us now take a look at $P(E_{k,\tilde{k}})$. By Lemma 2.2.9 we must estimate $B_{\mathbb{H}}(\mathcal{A},T)$, which represent the number of vectors $\left(k_j^{(p)},\ell_j^{(p)}\right)\in T^{2Km}$ that satisfy $\sum_{(r,p)}(k_r^{(p)}-k_{r-1}^{(p)})=0$ and $\sum_{(s,p)}(k_{s+1}^{(p)}\pm k_s^{(p)})=0$ for all $A\in\mathcal{A}$ with $\mathcal{A}\in P(2Km,t)$. Since we have t independent linear constraints $B_{\mathbb{H}}(\mathcal{A},T)$ is bounded from above by $|T|^{2Km-t}\le M^{2Km-t}$. Thus, by taking again $\theta=\frac{N}{M}$, we get

$$\mathbb{E}_X\left[\left|((HR_T)^m\operatorname{sgn}c)_{k\tilde{k}}\right|^{2K}\right]\le\sum_{t=1}^{Km}N^tS_2(2Km,t)M^{2Km-t}=\;M^{2Km}F_{2Km}(\theta),$$

and, therefore,

$$P(E_{k,\tilde{k}}) = \beta^{-2n} \sum_{m=1}^{n} \theta_{2mK_m} F_{2mK_m}(\theta) = \beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta).$$

Let us denote by $\mathbb{P}$(failure) the probability that exact reconstruction of f using ℓ_1-minimization fails.

By Lemma 2.2.4 and the above estimates it follows

$$\mathbb{P}(\text{failure}) \leq \mathbb{P}\left(\{\mathcal{F}_{TX}\{\text{is not injective}\} \cup \{\sup_{(k,\tilde{k})\in T^c} |P_{k,\tilde{k}}| \geq 1\}\right)$$

$$\leq \sum_{(k,\tilde{k})\in[-\rho,\rho]^2\cap\mathbb{Z}^2} \mathbb{P}(E_{k,\tilde{k}}) + \mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\| \geq \kappa\right) \leq D\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2}\, M\, G_{2n}(\theta)$$

under the conditions

$$a_1 = a = \sum_{m=1}^{n} \beta^{n/K_m} < 1, \quad a_2 + a_1 = 1, \text{ i.e., } a_2 = 1 - a,$$

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1} M^{-3/2} = \frac{1-a}{1+a} M^{-3/2}. \square$$

Let us remark that, given n, it is reasonable to take $K_m \approx m/n,\ m = 1, \ldots, n$ rounding m/n to the nearest integer. Then β is chosen quite close to the maximal value such that $a = \sum_{m=1}^{n} \beta^{n/K_m} < 1$. By our choice of K_m we approximately have

$$\sum_{m=1}^{n} \beta^{n/K_m} \approx \sum_{m=1}^{n} \beta^m \approx \frac{\beta}{1-\beta}.$$

Thus, the optimal β will always be close to $1/2$.

2.2.7 Proof of Theorem 2.2.2

As for bicomplex settings, the proof of the Theorem 2.2.2 for quaternions settings is the same as in [69], because it depends only on set partition (see Appendix A) without entering the algebraic structure of quaternions numbers.

2.2.8 Proof of Theorem 2.2.3

To prove our theorem we will need some modifications of the results obtained from the Lemma 2.2.8 and Lemma 2.2.9. For each one we will use the expression already defined in (2.38) and (2.41) and denoted it by $C_{\mathbb{H}}(\mathcal{A}, T)$ and $B_{\mathbb{H}}(\mathcal{C}, T)$, respectively. For this propose, similarly as [69] we consider T as a random set modeled by (2.8). We will start first with (2.38) as follows.

Lemma 2.2.10. *For $\mathcal{A} \in P(2n,t)$ it holds*

$$\mathbb{E}\left[C_{\mathbb{H}}(\mathcal{A},T)\right] \leq \sum_{s=2}^{n} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} D^{-R} \#\{\mathcal{B} \in U(2n,s),\ \operatorname{rank} M(\mathcal{A},\mathcal{B}) = R\}.$$

Proof. Using linearity of expectation value we obtain

$$\begin{aligned}
\mathbb{E}\left[C_{\mathbb{H}}(\mathcal{C},T)\right] = \mathbb{E}\Bigg[& \sum_{\substack{(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2 \\ (k_j,\tilde{k}_j) \neq (k_{j+1},\tilde{k}_{j+1})}} \prod_{j=1}^{2n} \mathbb{1}_{\{(k_j,\tilde{k}_j)\in T\}} \\
& \times \prod_{A\in\mathcal{A}} \delta\left(\sum_{r\in A}(k_{r+1}-k_r)\right) \delta\left(\sum_{r\in A}(\tilde{k}_{r+1} \pm \tilde{k}_r)\right) \Bigg] \\
= & \sum_{\substack{(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2 \\ (k_j,\tilde{k}_j) \neq (k_{j+1},\tilde{k}_{j+1})}} \mathbb{E}\left[\prod_{j=1}^{2n} \mathbb{1}_{\{(k_j,\tilde{k}_j)\in T\}}\right] \\
& \times \prod_{A\in\mathcal{A}} \delta\left(\sum_{r\in A}(k_{r+1}-k_r)\right) \delta\left(\sum_{r\in A}(\tilde{k}_{r+1} \pm \tilde{k}_r)\right).
\end{aligned}$$

Hereby, $\mathbb{1}_{\{(k,\tilde{k})\in T\}}$ denotes an indicator variable which is 1 if and only if $(k,\tilde{k}) \in T$. The expression $\mathbb{E}\left[\prod_{j=1}^{2n} \mathbb{1}_{\{(k,\tilde{k})\in T\}}\right]$ depends on how many different $(k_j,\tilde{k}_j)$'s there are. In this way, once again partitions come into play. If $(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2)^{2n}$ is a vector satisfying $(k_j,\tilde{k}_j) \neq (k_{j+1},\tilde{k}_{j+1})$ then we associate a partition $\mathcal{B} = (B_1,\ldots,B_s)$ of $\{1,\ldots,2n\}$ such that j and j' are in the same set B_i if and only if $(k_j,\tilde{k}_j) = (k_{j'},\tilde{k}_{j'})$. Obviously, j and $j+1$ must be contained in different blocks for all j due to the condition $(k_j,\tilde{k}_j) \neq (k_{j+1},\tilde{k}_{j+1})$ (once again we agree on the convention that $2n+1$ is identified with 1). In other words $\mathcal{B}$ has no adjacencies, i.e., $\mathcal{B} \in U(2n,s)$. Now if $\mathcal{B}$ has $|\mathcal{B}| = s$ blocks then by using the probability model $\mathbb{P}((k,\tilde{k}) \in T) = \tau$ (see (2.8)) for T and stochastic independence

$$\mathbb{E}\left[\prod_{i=1}^{2n} \mathbb{1}_{\{(k_i,\tilde{k}_i)\in T\}}\right] = \mathbb{E}\left[\prod_{j=1}^{s} \mathbb{1}_{\{(k,\tilde{k})_{B_j}\in T\}}\right] = \prod_{j=1}^{s} \mathbb{E}\left[\mathbb{1}_{\{(k,\tilde{k})_{B_j}\in T\}}\right] = \tau^s, \tag{2.43}$$

where (unambiguously) $(k,\tilde{k})_{B_j} = (k,\tilde{k})_i$ if $i \in B_j$. We further introduce the notation $\sigma_{\mathcal{B}}(r) = j$ if and only if $r \in B_j \in \mathcal{B}$. This leads us to

$$\begin{aligned}
& \mathbb{E}\left[C_{\mathbb{H}}(\mathcal{C},T)\right] \\
= & \sum_{s=2}^{n} \tau^s \sum_{\mathcal{B}\in U(2n,s)} \sum_{\substack{(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2 \\ (k_i,\tilde{k}_i)\text{ p.w. different}}} \prod_{A\in\mathcal{A}} \delta\left(\sum_{r\in A}(k_{\sigma_{\mathcal{B}}(r+1)} - k_{\sigma_{\mathcal{B}}(r)})\right).
\end{aligned}$$

Clearly, the expressions $\prod_{A\in\mathcal{A}} \delta\left(\sum_{r\in A}(k_{\sigma_{\mathcal{B}}(r+1)} - k_{\sigma_{\mathcal{B}}(r)})\right)$ is 1 and $\prod_{A\in\mathcal{A}} \delta\left(\sum_{r\in A}(\tilde{k}_{\sigma_{\mathcal{B}}(r+1)} - \tilde{k}_{\sigma_{\mathcal{B}}(r)})\right)$ is 1 if and only if, respectively,

$$\sum_{r\in A}(k_{\sigma_{\mathcal{B}}(r+1)} - k_{\sigma_{\mathcal{B}}(r)}) = 0, \text{ for all } A \in \mathcal{A} \tag{2.44}$$

and

$$\sum_{r\in A}(\tilde{k}_{\sigma_{\mathcal{B}}(r+1)} \pm \tilde{k}_{\sigma_{\mathcal{B}}(r)}) = 0, \text{ for all } A \in \mathcal{A} \tag{2.45}$$

and 0 otherwise. For $j \in \{1,\dots,s\}$ the term $(k_j, \tilde{k}_j)$ appears $|A_i \cap B_j|$ times as $(k_{\sigma_{\mathcal{B}}(r)}, \tilde{k}_{\sigma_{\mathcal{B}}(r)})$ when r runs through $A_i \in \mathcal{A}$. Let $M = M(\mathcal{A},\mathcal{B})$ denote the $t\times s$ matrix whose entries are defined by (2.5). Then (2.44) and (2.45) is satisfied if and only if $(k_1,\tilde{k}_1),\dots,(k_s,\tilde{k}_s) \in ([-\rho,\rho]^2)^s$ is contained in the kernel of $M(\mathcal{A},\mathcal{B})$. Thus, if the rank of $M(\mathcal{A},\mathcal{B})$ equals R then the number of vectors $(k_1,\tilde{k}_1),\dots,(k_s,\tilde{k}_s) \in ([-\rho,\rho]^2)^s$ for which (2.44) and and (2.45) is satisfied can be bounded by D^{s-R} where $D = (2\rho+1)^2$. (Here we even neglected the condition that the $(k_1,\tilde{k}_1),\dots,(k_s,\tilde{k}_s)$ should be pairwise different). So finally we obtain

$$\begin{aligned}
\mathbb{E}\left[C_{\mathbb{H}}(\mathcal{C},T)\right] &\le \sum_{s=2}^{n} \tau^s \sum_{R=0}^{\min\{s,t\}-1} D^{s-R}\#\{\mathcal{B}\in U(n,s),\ \mathrm{rank}\, M(\mathcal{A},\mathcal{B}) = R\} \\
&= \sum_{s=2}^{n} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} D^{-R}\#\{\mathcal{B}\in U(n,s),\ \mathrm{rank}\, M(\mathcal{A},\mathcal{B}) = R\},
\end{aligned}$$

where we substituted $\mathbb{E}|T| = \tau D$. □

Since $E = E_X E_T$ by Fubini's theorem and stochastic independence of T and X the previous result yields together with Lemma 2.2.8

$$\begin{aligned}
&\mathbb{E}\left[\|H_0^n\|_F^2\right] \\
&\le \sum_{t=1}^{\min\{n,N\}} \frac{N!}{(N-t)!} \sum_{\mathcal{A}\in P(2n,t)} \sum_{s=2}^{n} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} D^{-R}\#\{\mathcal{B}\in U(n,s),\ \mathrm{rank}\, M(\mathcal{A},\mathcal{B}) = R\} \\
&\le \sum_{t=1}^{\min\{n,N\}} \frac{N!}{(N-t)!} \sum_{s=2}^{n} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} D^{-R} Q(2n,t,s,R) = N^{2n} W(n,N,\mathbb{E}|T|,D)
\end{aligned}$$

by definition (A.4), in Appendix A, of the numbers $Q(2n,t,s,R)$ and by definition (2.9) of the function W. Markov's inequality yields

$$\mathbb{P}(\left\|(N^{-1}H_0)^n\right\|_F \ge \kappa) \le N^{-2n}\kappa^{-2}\mathbb{E}[\|H_0^n\|_F^2] \le \kappa^{-2} W(n,N,\mathbb{E}|T|,D).$$

We remark that by the same argument as in the proof of Theorem 2.2.3 $\mathcal{F}_{TX}$ is injective in the event $\left\|(N^{-1}H_0)^n\right\|_F \le 1$.

Let us turn now to the estimation of $\mathbb{P}(E_{k,\tilde{k}})$. From Lemma 2.2.9 one realizes that we need to estimate the expected value of $B_{\mathbb{H}}(\mathcal{C},T)$ defined in (2.41) .

Lemma 2.2.11. *For $\mathcal{A} \in P(2Km,t)$ it holds*

$$\mathbb{E}\left[B_{\mathbb{H}}(\mathcal{C},T)\right] \leq \sum_{s=1}^{2Km} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}-1} D^{-R}\#\{\mathcal{B} \in U^*(2K,m,s),\ \mathrm{rank}\, L(\mathcal{A},\mathcal{B}) = R\}.$$

Proof. As in the proof of the previous lemma we may write

$$\mathbb{E}\left[B_{\mathbb{H}}(\mathcal{C},T)\right] = \sum_{\substack{(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2 \\ (k_j,\tilde{k}_j) \neq (k_{j+1},\tilde{k}_{j+1})}} \mathbb{E}\left[\prod_{(p,j)\in[2K]\times[m]} \mathbb{1}_{\{(k,\tilde{k})\in T\}}\right]$$
$$\times \prod_{A\in\mathcal{A}} \delta\left(\sum_{(r,p)\in A} \alpha(k_{r+1}^{(p)} - k_r^{(p)})\right) \delta\left(\sum_{(r,p)\in A} \alpha(\tilde{k}_{r+1}^{(p)} - \tilde{k}_r^{(p)})\right).$$

Once again $\mathbb{E}\left[\prod_{(p,j)\in[2K]\times[m]} \mathbb{1}_{\{(k_j^{(p)},\tilde{k}_j^{(p)})\in T\}}\right]$ depends on how many different $(k^{(p)},\tilde{k}_r^{(p)})$'s there are. So if $\left((k_1^{(1)},\tilde{k}_1^{(1)}),\ldots,(k_m^{(2K)},\tilde{k}_m^{(2K)})\right) \in ([-\rho,\rho]^2)^{(2Km)}$ is a vector satisfying

$$\left(k_j^{(p)},\tilde{k}_j^{(p)}\right) \neq \left(k_{j+1}^{(p)},\tilde{k}_{j+1}^{(p)}\right) \text{ for all } j\in[m],\ p\in[2K] \tag{2.46}$$

then we associate a partition $\mathcal{B} = (B_1,\ldots,B_s)$ of $[2K]$ such that (p,j) and (p',j') are contained in the same block if and only if $k_j^{(p)} = k_{j'}^{(p')}$. Obviously, (p,j) and $(p,j+1)$ cannot be contained in the same block due to the condition (2.46). In other words, $\mathcal{B}$ belongs to $U^*(2K,m,s)$. Now, if $\mathcal{B}$ has s blocks, i.e., there are s different values of $(k_j^{(p)},\tilde{k}_j^{(p)})$, then

$$\mathbb{E}\left[\prod_{(p,j)\in[2K]\times[m]} \mathbb{1}_{\{(k_j^{(p)},\tilde{k}_j^{(p)})\in T\}}\right] = \tau^s$$

as in (2.43). Once more, we use the notation $\sigma_{\mathcal{B}}(p,j) = i$ if $(p,j)\in B_i\in\mathcal{B}$ and $\sigma(p,0) = 0$. (Recall that by definition $(k_0^{(p)},\tilde{k}_0^{(p)}) = (k_0,\tilde{k}_0) = (k,\tilde{k})$.) Thus,

$$\mathbb{E}\left[B_{\mathbb{H}}(\mathcal{C},T)\right]$$
$$= \sum_{s=1}^{2n} \tau^s \sum_{\mathcal{B}\in U^*(2K,m,s)} \sum_{\substack{(k_1,\tilde{k}_1),\ldots,(k_{2n},\tilde{k}_{2n}) \in [-\rho,\rho]^2 \\ (k_i,\tilde{k}_i) \text{ p.w. different}}}$$
$$\prod_{A\in\mathcal{A}} \delta\left(\sum_{(p,j)\in A} \alpha(k_{\rho_{\mathcal{B}}(p,j+1)} - k_{\rho_{\mathcal{B}}(p,j)})\right) \delta\left(\sum_{(p,j)\in A} \alpha(\tilde{k}_{\rho_{\mathcal{B}}(p,j+1)} \pm \tilde{k}_{\rho_{\mathcal{B}}(p,j)})\right).$$

The term $\prod_{A\in\mathcal{A}} \delta\left(\sum_{(p,j)\in A} \alpha(\tilde{k}_{\rho_{\mathcal{B}}(p,j+1)} - \tilde{k}_{\rho_{\mathcal{B}}(p,j)})\right)$ and $\prod_{A\in\mathcal{A}} \delta\left(\sum_{(p,j)\in A} \alpha(\tilde{k}_{\rho_{\mathcal{B}}(p,j+1)} \pm \tilde{k}_{\rho_{\mathcal{B}}(p,j)})\right)$ contributes to the sum if and only if, respectively,

$$\sum_{(p,j)\in A} \alpha(k_{\rho_{\mathcal{B}}(p,j+1)} - k_{\rho_{\mathcal{B}}(p,j)}) = 0 \text{ for all } A \in \mathcal{A}$$

and

$$\sum_{(p,j)\in A} \alpha(\tilde{k}_{\rho_{\mathcal{B}}(p,j+1)} \pm \tilde{k}_{\rho_{\mathcal{B}}(p,j)}) = 0 \text{ for all } A \in \mathcal{A}.$$

By definition (A.5) (in Appendix A) of the matrix $L(\mathcal{A},\mathcal{B})$ and since $(k_0,\tilde{k}_0) = (k,\tilde{k})$ this is equivalent to

$$L(\mathcal{A},\mathcal{B})\left((k_1,\tilde{k}_1),\ldots,(k_s,\tilde{k}_s)\right)^T = kv(\mathcal{A},\mathcal{B}), \tag{2.47}$$

where $v = v(\mathcal{A},\mathcal{B})$ is the t-dimensional vector with entries

$$v_i = \sum_{(p,1)\in A_i} \alpha^{(p)}, \quad i = 1,\ldots,t.$$

(If $d > 1$ then (2.47) has to interpreted vector-valued, i.e., for each component of $(k,\tilde{k}) \in [-\rho,\rho]^2$ and of $(k_1,\tilde{k}_1),\ldots,(k_s,\tilde{k}_s) \in [-\rho,\rho]^2$ we have one equation with the same $L(\mathcal{A},\mathcal{B})$ and the same $v(\mathcal{A},\mathcal{B})$.) If the rank of $L(\mathcal{A},\mathcal{B})$ equals R then we can bound the number of solutions to (2.47) by D^{s-R}. Hence, we obtain the bound

$$\mathbb{E}\left[B(\mathcal{A},T)\right] \leq \sum_{s=1}^{2Km} \tau^s \sum_{R=0}^{\min\{s,t\}} D^{s-R} \#\{\mathcal{B} \in U^*(2K,m,s),\ \operatorname{rank} L(\mathcal{A},\mathcal{B}) = R\}.$$

Since $\mathbb{E}|T| = \tau D$ this proves the lemma. $\square$

Using the Lemma 2.2.9 the previous result yields

$$\begin{aligned}\mathbb{E}\left[|((HR_T)^m\sigma)_{k,\tilde{k}}|^{2K}\right] &\leq \sum_{t=1}^{\min\{Km,N\}} \frac{N!}{(N-t)!} \sum_{s=1}^{2Km} (\mathbb{E}|T|)^s \sum_{R=0}^{\min\{s,t\}} Q^*(2K,m,t,s,R) D^{-R} \\ &= N^{2Km} Z(K,m,N,\mathbb{E}|T|,D)\end{aligned}$$

where $Q^*(2K,m,t,s,R)$ are the numbers defined in (A.6), in Appendix A. From (2.40) follows

$$\mathbb{P}(E_{k,\tilde{k}}) \leq \beta^{-2n} \sum_{m=1}^{n} Z(K_m,m,N,\mathbb{E}|T|,D).$$

Finally, let $\mathbb{P}(\text{failure})$ denote the probability that exact reconstruction of f fails. By Lemma 2.2.4 and the expressions (2.28) and (2.29) using the fact $\{\mathcal{F}_{TX} \text{ is not injective}\} \subset \{\left\|(N^{-1}H_0)^n\right\|_F \geq \kappa\}$ we finally obtain

$$\mathbb{P}(\text{failure}) \leq \mathbb{P}\left(\{\mathcal{F}_{TX}\{\text{is not injective}\} \cup \{\sup_{(k,\tilde{k})\in T^c} |P_{k,\tilde{k}}| \geq 1\}\right)$$

$$\leq \sum_{(k,\tilde{k})\in[-\rho,\rho]^2} \mathbb{P}(E_{k,\tilde{k}}) + \mathbb{P}\left(\left\|\left(N^{-1}H_0\right)^n\right\| \geq \kappa\right) + \mathbb{P}(|T| \geq (\alpha+1)\mathbb{E}|T|)$$

$$\leq D\beta^{-2n}\sum_{m=1}^{n} Z(K_m, m, N, \mathbb{E}|T|, D) + \kappa^{-2}W(n, N, \mathbb{E}|T|, D) + \exp\left(-\frac{3\alpha^2}{6+2\alpha}\mathbb{E}|T|\right)$$

under the conditions (see also (2.27))

$$a_1 = a = \sum_{m=1}^{n} \beta^{n/K_m} < 1, \quad a_2 + a_1 = 1, \text{ i.e., } a_2 = 1 - a,$$

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}((\alpha+1)\mathbb{E}|T|)^{-3/2} = \frac{1-a}{1+a}((\alpha+1)\mathbb{E}|T|)^{-3/2}.$$

This proves Theorem 2.2.3.

2.3 Applications

Although there exists a large variety of applications of quaternionic signals, we will restrict ourselves to the case of color-encoded images represented by quaternions in this section, i.e. the R-,G-, and B-components of the image are represented by the **i**-, **j**-, **k**-components of a given quaternion, respectively. In what follows we present some numerical experiments. Namely, we reconstruct a given signal by means of the ℓ_1-minimization problem

$$\min \|x\|_{\ell_1} = \sum_{s=1}^{N} |x_s|$$
$$\text{s.t. } \Phi x = y, \tag{2.48}$$

where $\Phi \in \mathbb{H}^{M\times N}$ is our quaternionic sampling matrix, $y \in \mathbb{H}^{M\times 1}$ is the vector consisting of chosen random pixels from the given image under the assumption that the signal vector $x \in \mathbb{H}^{N\times 1}$ is k-sparse, i.e., at most k entries of x are non-zero. The support of x is the set of indexes corresponding to the non-zero entries, $\operatorname{supp} x = \{s \in \{1,\ldots,N\} : x_s \neq 0\}$. We aim to illustrate that an effective signal reconstruction is possible even if using only a small amount of the signal information.

For the implementation of our ℓ_1-minimization algorithm we use the Matlab toolbox ℓ1-Magic [18].

A quaternion is defined as follows (see for instance [48])

$$q = R(q) + \mathbf{i}I(q) + \mathbf{j}J(q) + \mathbf{k}K(q)$$

where $R(q)$ is the real part of the quaternion and $I(q), J(q)$ and $K(q)$ are its three imaginary components. In a similar way, we have for vectors and matrices of quaternions the decomposition

$$\mathbf{v} \;=\; R(\mathbf{v}) + \mathbf{i}I(\mathbf{v}) + \mathbf{j}J(\mathbf{v}) + \mathbf{k}K(\mathbf{v})$$

$$\mathbf{M} \;=\; R(\mathbf{M}) + \mathbf{i}I(\mathbf{M}) + \mathbf{j}J(\mathbf{M}) + \mathbf{k}K(\mathbf{M}).$$

Since $\ell 1$-Magic works only with real-valued vectors we need to modify our quaternionic-valued system. We rewrite the quaternionic multiplication $(\alpha + \mathbf{i}\beta + \mathbf{j}\gamma + \mathbf{k}\delta)(x + \mathbf{i}y + \mathbf{j}v + \mathbf{k}w) = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d.$ as a matrix-vector multiplication, i.e.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & -\gamma & \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ v \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

This allows us to rewrite the quaternionic linear system (2.48) in the form $\tilde{y} = \mathcal{M}\tilde{x}$ with

$$\mathcal{M} = \begin{pmatrix} R(\Phi) & -I(\Phi) & J(\Phi) & -K(\Phi) \\ -I(\Phi) & R(\Phi) & -K(\Phi) & J(\Phi) \\ -J(\Phi) & K(\Phi) & R(\Phi) & -I(\Phi) \\ -K(\Phi) & -J(\Phi) & I(\Phi) & R(\Phi) \end{pmatrix},$$

$$\tilde{x} = \begin{pmatrix} R(x) \\ I(x) \\ J(x) \\ K(x) \end{pmatrix}, \text{ and } \quad \tilde{y} = \begin{pmatrix} R(y) \\ I(y) \\ J(y) \\ K(y) \end{pmatrix}.$$

In our implementation we use the sampling matrix in its explicit form. This leads to large requirements in terms of memory since we deal with a matrix of dimension $M \times N^2$. To be able to work with such large images, we used a strategy of dividing the image into 8×8-blocks followed by an individual reconstruction of each block. The sampling matrices for the blocks themselves are constructed by using DCT (Discrete Cosine Transform) and DST (Discrete Sine Transform). Since we studied each block individually, we obviously got the additional problem of reassemble them. As our main objective is to show the feasibility of our approach as a proof-of-concept we decided to ignore that problem. Nevertheless, we would like to remark that a correct assembling procedure would require the use of the correct boundary condition for the reconstruction.

We performed the calculations on a computer with Intel(R) Core(TM) i7-4790U CPU 3.60 GHz, RAM 16GB, Windows 8.1, OS 64-bit(win64) and running Matlab R2012b.

For our examples we choose the following images: Lena (Figure 2.1), Galaxy (Figure 2.3), and the Saturn rings (Figure 2.5), each with $N^2 = 262144$ pixels (512×512). For the reconstruction we use 40000 pixels ($M = 625$ samples in each block) which corresponds to $\approx 15.26\%$ of the total information. The reconstructed images can be seen in Figure 2.2, Figure 2.4, and Figure 2.6, respectively.

Figure 2.1: The original image

Figure 2.2: The reconstructed image.

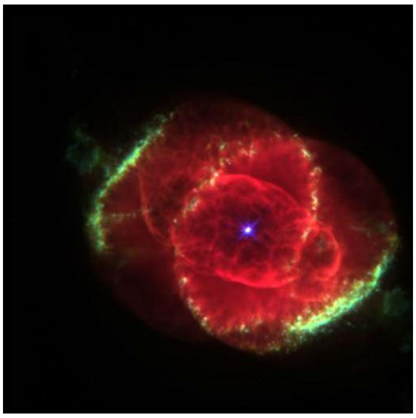

Figure 2.3: The original image

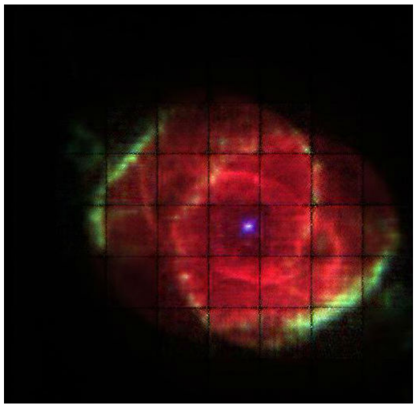

Figure 2.4: The reconstructed image

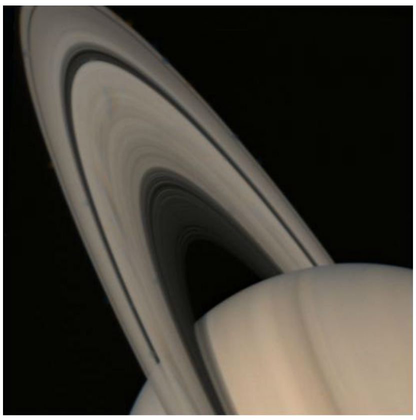

Figure 2.5: The original image

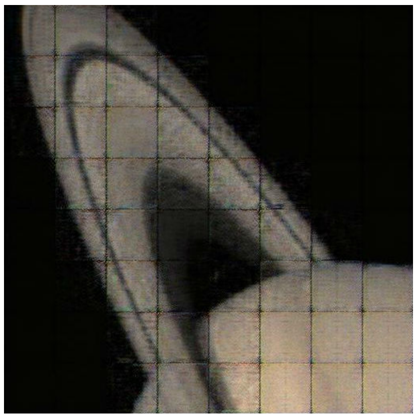

Figure 2.6: The reconstructed image

Another particular example is the chips image (Figure 2.7). This image combines textures and sharp edges. While the image is still blurred (as expected) it still contains all relevant details. For this example, the image had $N^2 = 65536$ pixels (256×256). For the reconstruction we use 10000 pixels ($M = 625$ samples in each block). Under the same conditions as above ($\approx 15.26\%$ of the original pixels where taken as random samples) we obtain our reconstruction (see Figure 2.8).

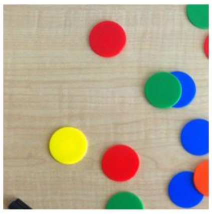

Figure 2.7: The original image

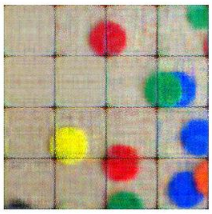

Figure 2.8: The reconstructed image

Chapter 3

The case of non-linear Fourier atoms or Takenaka-Malmquist system

> *We are servants rather than masters in mathematics.*
>
> Charles Hermite,

In this chapter we apply the approach to the case of the Takenaka-Malmquist system as a realization of an expansion in terms of non-linear Fourier atoms. We shall conclude this chapter with some applications of this scheme to some functions.

3.1 Non-linear Fourier atoms

Nonlinear Fourier atoms are a family of nonlinear Fourier bases, seen as an extension of the classical Fourier basis, that have been constructed and applied to signal processing [24, 29]). For any complex number $a = re^{it_a}$, $r = |a| < 1$, the nonlinear phase function $\theta_a(t)$ is defined by the radical boundary value of the Möbius transformation

$$\tau_a(z) = \frac{z-a}{1-\bar{a}z},$$

that is, the nonlinear Fourier atom is given by

$$e^{i\theta_a(t)} := \tau_a(e^{it}) = \frac{e^{it}-a}{1-\bar{a}e^{it}}.$$

Note that $\theta_a(t+2\pi) = \theta_a(t) + 2\pi$ and its derivative is the Poisson kernel

$$\theta_a'(t) = \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} = \mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right), \tag{3.1}$$

which satisfies

$$0 < \frac{1-r}{1+r} \leq \theta_a'(t) \leq \frac{1+r}{1-r}. \tag{3.2}$$

For any sequence $\{c_k\}_{k\in\mathbb{Z}}$ of finite nonzero terms holds

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} \left|c_k e^{ikx}\right|^2 dx = \frac{1}{2\pi}\int_{\mathbb{T}} \left|c_k e^{ik\theta_a(t)}\right|^2 \theta_a'(t)dt,$$

which by combining with (3.2) implies that we can consider the so-called nonlinear Fourier basis $\{e^{in\theta_a(t)}\}_{n\in\mathbb{Z}}$, in which $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle ([24], [68]). Note that if $a = 0$, $\{e^{in\theta_a(t)}\}_{n\in\mathbb{Z}}$ reduces to the classic Fourier basis $\{e^{int}\}_{n\in\mathbb{Z}}$. These atoms are star-like functions, convex with positive phase derivative on the boundary and they are linked to TM systems (see, for instance, [67]).

3.1.1 Hardy spaces

We consider the following function spaces: $\mathcal{L}^2(\mathbb{T})$ the Hilbert space of square integrable functions on the unit circle.

For $1 \leq p < \infty$ the Hardy space $\mathcal{H}^p$ is defined as the space of all analytic functions f in $\mathbb{D}$ for which the norm

$$\|f\|_p = \sup_{0\leq r<1}\left(\frac{1}{2\pi}\int_{[0,2\pi]} |f(re^{it})|^p dt\right)^{1/p}$$

if finite ([57, 39]). The space $\mathcal{H}^\infty$ consists of all bounded analytic functions f in $\mathbb{D}$ and the norm is now

$$\|f\|_\infty \sup_{|z|<1} |f(z)|.$$

For functions in $\mathcal{H}^p(\mathbb{D})$, $1 \leq p \leq \infty$, the radial limit

$$\tilde{f}(e^{it}) = \lim_{r\to 1} f(re^{it})$$

exists almost everywhere in t (Fatou's theorem), and indeed $\tilde{f} \in \mathcal{L}^p(\mathbb{T})$. Moreover

$$\|f\|_p = \sup_{0\leq r<1}\left(\frac{1}{2\pi}\int_{[0,2\pi]} |f(re^{it})|^p dt\right)^{1/p} =: \left\|\tilde{f}\right\|_{\mathcal{L}^p(\mathbb{T})}.$$

We normally identify f with $\tilde{f}$, and can regard $\mathcal{H}^p$ as the subspace of those functions in $\mathcal{L}^p(\mathbb{T})$ for which the negative Fourier coefficients vanish, that is:

$$\frac{1}{2\pi}\int_{[0,2\pi]} \tilde{f}(e^{it})e^{-int}dt = 0$$

for all $n < 0$. Then a function $\tilde{f} \sim \sum_{n=0}^{\infty} a_n z^n$ can be naturally identified with the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, defining an analytic function f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disk in the complex plane.

One can also obtain the extension from $\tilde{f}$ to f by convolving with the Poisson kernel K_r, namely

$$f(re^{it}) = \frac{1}{2\pi}\int_{[0,2\pi]} K_{r,t}(\theta - t)\tilde{f}(e^{it})dt$$

where $K_{r,t}(\theta - t)$ is Poisson kernel, from (3.1).

The case $p = 2$ is simpler, since for a function $f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$ we have

$$\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}.$$

We have the following inclusions:

$$\mathcal{H}^\infty \subset \mathcal{H}^p \subset \mathcal{H}^q \subset \mathcal{H}^1,$$

for $1 \leq q \leq p \leq \infty$.

3.1.2 Takenaka-Malmquist system as non-linear Fourier atoms

The Takenaka-Malmquist (TM) system belongs to the families of unit analytic signals with nonlinear phase and are closely linked to non-linear Fourier atoms.

With a given sequence $\{a_k\}_{k=1}^{\infty}$ of points in $\mathbb{D}$ we associate the modified Blaschke products $B_1, B_2 \ldots$, defined by

$$\mathcal{B}_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z}\varphi_n(z) \quad \text{and} \quad \mathcal{B}_{-n}(z) = \overline{\mathcal{B}_n\left(1/\bar{z}\right)},$$

for $n = 0, 1, 2, \ldots$ with the finite Blaschke product

$$\varphi_p(z) = \prod_{k=0}^{p-1} \frac{z - a_k}{1 - \bar{a}_k z}, \tag{3.3}$$

where $a_k \in \mathbb{D}$ for all $k \geq 0$ and $a_0 = 0$. Although the usual orthonormal Fourier atoms $\{e^{in\omega}\}_{n\in\mathbb{Z}}$ are complete in $\mathcal{L}^2(\mathbb{T})$ in general the nonlinear Fourier atoms $\{e^{in\theta(t)}\}$ are not orthogonal. The Gram-Schmidt orthonormalization process leads to the TM basis $\{\mathcal{B}_n \,:\, n \geq 0\}$ which is known to be complete in $\mathcal{H}^2(\mathbb{D})$ and the basis $\{\mathcal{B}_n \,:\, n \in \mathbb{Z}\}$ which is complete in $\mathcal{L}^2(\mathbb{T})$ if and only if it satisfies the condition

$$\sum_{n=1}^{\infty} (1 - |a_n|) = \infty. \tag{3.4}$$

The condition (3.4) implies that the Blaschke products φ_p in (3.3) converges to 0. We recall that if $a_k = 0$ we obtain the classical Fourier basis as a usual case (see, for instance [15]).

The inner product of two complex function g_1 and g_2 in $\mathbb{T}$ is defined as

$$\langle g_1, g_2\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}} \overline{g_1(z)}g_2(z)\frac{dz}{z}. \tag{3.5}$$

Alternatively from (3.5), the inner product can be written

$$\langle g_1, g_2 \rangle \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} \overline{g_1(e^{ix})} g_2(e^{ix}) dx$$

and the induced norm will be denoted by $\| \cdot \|_2$.

For instance, as the special case, when $a_n = b$ $(n \in \mathbb{N}^*)$, we have $\mathcal{B}_n = L_n^b$ $(n \in \mathbb{N}^*)$ is the discrete Laguerre system, and if $a_{2k-1} = a, a_{2k} = b$ $(k \in \mathbb{N}^*)$, then $(\mathcal{B}_n, n \in N^*)$ is the Kautz-system, investigated in [10].

3.2 Sparse sampling in Takenaka-Malmquist system

Since we want to follow again the approach of Rauhut [69] we have to look at the corresponding setting with trigonometric polynomials being replaced by atoms of the form

$$\mathcal{B}_{a_n}(x) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a}_n e^{ix}} \prod_{k=0}^{n-1} \frac{e^{ix} - a_k}{1 - \overline{a}_k e^{ix}},$$

where $a_n \in \mathbb{C}$ is such that $|a_n| < 1$. These atoms are star-like functions, convex with positive phase derivative on the boundary and they are linked to TM systems insofar as they represent elements of the orthogonal basis generated by $(a_0, a_1, \ldots, a_n, \ldots)$. Hereby, we again denote by $\prod_d$ the space spanned by at most d elements; in other words, an element $f \in \prod_d$ is of the form

$$f(x) = \sum_{n=1}^{d} c_n \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a}_n e^{ix}} \prod_{k=0}^{n-1} \frac{e^{ix} - a_k}{1 - \overline{a}_k e^{ix}}, \qquad x \in [0, 2\pi],\ |a_k| < 1,\ c_n \in \mathbb{C}. \tag{3.6}$$

We assume that the sequence of coefficients $c = (c_k)$ is supported on a set T which is much smaller than the dimension of $\prod_d$, that is to say, the finite combination in (3.6) is sparse. However, as in the previous cases a priori nothing is known about T apart from its maximum size. Thus, it is useful to introduce the set (not a linear space) $\prod_d(M) \subset \prod_d$ of all polynomials of type (3.6) such that their sequence of coefficients $c = (c_n)$ have support on a set $T \subset \{1, \ldots, d\}$ satisfying $|T| \leq M$, i.e., $f \in \prod_d(M)$ is of the form

$$f(x) = \sum_{n \in T} c_n \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a}_n e^{ix}} \prod_{k=0}^{n-1} \frac{e^{ix} - a_k}{1 - \overline{a}_k e^{ix}}.$$

Again, the objective is, given a sampling set $X := \{x_1, x_2, \ldots, x_N\}$ of independent random variables having uniform distribution on $[0, 2\pi]$, to reconstruct $f \in \prod_d(M)$ from the samples $f(x_j)$ at those N randomly chosen points.

3.2.1 Description of the main results

The next theorems are the analogues of the main theorems from the previous chapter.

Theorem 3.2.1. *Assume $f \in \prod_q(M)$ with some sparsity $M \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_N \in [0, 2\pi]$ be independent random variables having the uniform distribution on $[0, 2\pi]$. Choose $n \in \mathbb{N}$, $\beta > 0$, $\kappa > 0$ and $K_1, \cdots, K_n \in \mathbb{N}$ such that*

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \text{ and } \frac{\kappa}{1-\kappa} \leq \frac{1-a}{1+a} M^{-3/2}. \tag{3.7}$$

Set $\theta := (N\mathcal{K})/M$. Then with probability at least

$$1 - \left(\mathcal{C}_D \left[|D| \beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2} \mathcal{K}^{2n} M\, N^{2n} G_{2n}(\theta) \right] \right) \tag{3.8}$$

f can be reconstructed exactly from its sample values $f(x_1), \ldots, f(x_N)$ by solving the minimization problem (3.10).

As already stated in the previous chapter given n it is reasonable to take $K_m \approx m/n$, $m = 1, \ldots, n$ rounding m/n to the nearest integer. Then β is chosen quite close to the maximal value such that $a = \sum_{m=1}^{n} \beta^{n/K_m} < 1$. By our choice of K_m we approximately have

$$\sum_{m=1}^{n} \beta^{n/K_m} \approx \sum_{m=1}^{n} \beta^m \approx \frac{\beta}{1-\beta}.$$

Thus, the optimal β will always be close to $1/2$.

Theorem 3.2.2. *There exists an absolute constant $C > 0$ such that the following is true. Assume $f \in \prod_d(M)$ for some sparsity $M \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_N \in [0, 2\pi]$ be independent random variables having the uniform distribution on $[0, 2\pi]$. If for some $\epsilon > 0$ it holds*

$$N \geq CM \log(|D|/\epsilon) \tag{3.9}$$

then with probability at least $1 - \epsilon$ the function f can be recovered from its sample values $f(x_j)$, $j = 1, \ldots, N$, by solving the ℓ_1-minimization problem

$$\min \|(c_n)\|_1 := \sum_{n \in D} |c_n|, \qquad s.a \quad g(x_j) := \sum_{n \in D} c_n \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} e^{ix_j}} \prod_{k=0}^{n-1} \frac{e^{ix_j} - a_k}{1 - \overline{a_k} e^{ix_j}}, j = 1, \ldots, N. \tag{3.10}$$

Theorem 3.2.3. *Let $x_1, x_2, \ldots, x_N \in [0, 2\pi]$ be independent random variables having the uniform distribution on $[0, 2\pi]$. Further assume that T is a random subset of $[0, 2\pi]$ modelled by*

$$a := \sum_{m=1}^{n} \beta^{n/K_m} < 1 \qquad \textit{and} \qquad \frac{k}{1-k} \leq \frac{1-a}{1+a} ((\alpha+1)\mathbb{E}|T|)^{-3/2}. \tag{3.11}$$

Then with probability at least

$$1-\Big(\kappa^{-2}W(n,N,\mathbb{E}|T|,|D|)+\beta^{-2n}|D|\sum_{m=1}^{n}Z(K_m,m,N,\mathbb{E}|T|,|D|)+\exp\big(-\frac{3\alpha^2}{6+2\alpha}\mathbb{E}|T|\big)\Big) \tag{3.12}$$

any $f\in\prod_T\subset\prod_q(|T|)$ can be reconstructed exactly from its sample values $f(x_1),\ldots,f(x_N)$ by solving the minimization problem (3.10).

Since we could already observe in the previous two chapters that the principal difficulty resides in the proof of the first theorem, since the adaptations for the proofs of the other two theorems do not depend on the actual choice of the system, i.e. the proof of the first theorem provides the necessary basis for the proof of the other two theorems without the need for further modifications.

3.2.2 Proof of main results

As before, we use some auxiliary notations: $\ell_2(D)$, $\ell_2(T)$, $\ell_2(X)$ will denote the ℓ_2-spaces of sequences indexed by $D=\{1,2,\cdots,d\}$, T and X, respectively, all endowed with the usual Euclidean norm. Moreover, we introduce the operator $\mathcal{F}_X:\ell_2(D)\to\ell_2(X)$ given as

$$\mathcal{F}_X:=\left[\frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}e^{ix_j}}\prod_{k=0}^{n-1}\frac{e^{ix_j}-a_k}{1-\overline{a_k}e^{ix_j}}\right]_{j=1,\ldots,N,\ n=1,\ldots,d}.$$

We recall that $|a_k|<1$, for all $k=1,\ldots,d$.

By $\mathcal{F}_{TX}$ we represent the restriction of $\mathcal{F}_X$ to sequences supported only on T, thus, an operator acting from $\ell_2(T)$ in $\ell_2(X)$. Also, we need their adjoint operators, $\mathcal{F}^*_X:\ell_2(X)\to\ell_2(D)$ and $\mathcal{F}^*_{TX}:\ell_2(X)\to\ell_2(T)$.

Our problem is to reconstructing a sequence $c\in\ell_2(D)$ from $\beta=\mathcal{F}_Xc\in\ell_2(X)$, by solving the problem

$$\min\|c\|_1\quad\text{subject}\quad\text{to}\quad\mathcal{F}_Xc=\beta.$$

Obviously, if $k\notin\operatorname{supp}c$ then $\operatorname{sgn}(c)_k=0$ while $|\operatorname{sgn}(c)_k|=1$ for all $k\in\operatorname{supp}c$.

Le us start with the usual basic lemma.

Lemma 3.2.4. *Let $c\in\ell_2(D)$ and $T:=\operatorname{supp}c$. Assume $\mathcal{F}_{TX}:\ell_2(T)\to\ell_2(X)$ to be injective. Suppose that there exists a vector $P\in\ell_2(D)$ with the following properties:*
(i) $P_k=\operatorname{sgn}(c)$ for all $k\in T$,
(ii) $|P_k|<1$ for all $k\notin T$,
(iii) there exists a vector $\lambda\in\ell_2(X)$ such that $P=\mathcal{F}^*_X\lambda$.

Then c is unique minimizer to the problem (3.10).

Proof. Let us assume $X\neq\emptyset$ and $c\neq0$ to exclude the trivial cases. Furthermore, let us suppose that the vector P exist. Let b be any vector different to c with $\mathcal{F}_Xb=\mathcal{F}_Xc$. Consider $h:=b-c$, then $\mathcal{F}_X q$ vanishes on X. This means that for b_k, $k \in T$, we have the following estimate

$$\begin{aligned} |b_k| &= |c_k + d_k| = |(c_k + h_k)\overline{\operatorname{sgn} c_k} \operatorname{sgn} c_k| \\ &= |\,(c_k\, \overline{\operatorname{sgn} c_k} + h_k\, \overline{\operatorname{sgn} c_k})\operatorname{sgn} c_k| \\ &= |\,|c_k| + h_k\, \overline{\operatorname{sgn} c_k}||\overline{\operatorname{sgn} c_k}| \\ &= |\,|c_k| + h_k\, \overline{\operatorname{sgn} c_k}| \geq |c_k| + \operatorname{Re}(h_k\, \overline{\operatorname{sgn} c_k}) \\ &= |c_k| + \operatorname{Re}\left(h_k\, \overline{P_k}\right). \end{aligned}$$

Thus, for any $k \in T$ we have $|c_k| + \operatorname{Re}\left(h_k\, \overline{P_k}\right) \leq |b_k|$. Otherwise, for $k \notin T$ we have $\operatorname{Re}\left(h_k\, \overline{P_k}\right) \leq |h_k| = |h_k|$ since $|P_k| < 1$. Thus

$$||b||_{\ell_1} \geq ||c||_{\ell_1} + \sum_{k \in [-q,q] \cap \mathbb{Z}} \operatorname{Re}\left(h_k\, \overline{P_k}\right).$$

Now, from condition (iii) we can conclude

$$\sum_{k \in [-q,q] \cap \mathbb{Z}} \operatorname{Re}\left(h_k\, \overline{P_k}\right) = \operatorname{Re}\left(\sum_{k \in [-q,q] \cap \mathbb{Z}} h_k\, \overline{\left(\mathcal{F}_X^* \lambda\right)_k}\right) = \operatorname{Re}\left(\sum_{i=1}^{N} \left(\mathcal{F}_X\, h\right)(x_i)\overline{\lambda(x_i)}\right) = 0$$

whereas $\mathcal{F}_X\, h$ vanishes, similarly in (2.13). Thus, $||b||_{\ell_1} \geq ||c||_{\ell_1}$. The equality holds when $||h_k|| = \operatorname{Re}\left(h_k\, \overline{P_k}\right)$ for all $k \notin T$. Since $||P_k|| < 1$, this forces h to vanish outside of T. Taking in account the injectivity of $\mathcal{F}_{TX}$ we have that since $\mathcal{F}_X h$ vanishes on X, h vanishes identically and we have $b = c$. Thus, this shows that c is unique minimizer $c^\sharp$ to the problem (3.10). □

Let us now start again our proof. The first part is basically the same as in the other two cases, but we will repeat it here for the sake of convenience.

Before we can start the proof in the usual way we need the following lemma. Let us point out that the usual estimates on the maximum of a Blaschke product which can be found in the literature are not good enough in this context since we want to estimate a probability.

Lemma 3.2.5. *Let be $\epsilon > 0$ and $a_n, z \in \mathbb{C}$ such that $a_n = \alpha + i\beta$ and $z = x + iy$, $n = 1, \ldots, d$. If $|a_n| < 1$, $|z| = 1$, and either $|\pm 1 - a_n| = \epsilon$ or $|\pm i - a_n| = \epsilon$ then the set of functions*

$$f_{a_n}(z) = \frac{1 - |a_n|^2}{|1 - \overline{a}_n z|^2}, \tag{3.13}$$

has a uniform maximum which can be estimated by

$$\max_{a_n \in \mathbb{D}} \max_{z_A \in \mathbb{T}} \frac{1 - |a_n|^2}{|1 - \overline{a}_n z|^2} \leq \frac{2}{\epsilon} - 1.$$

Proof. Let us take $z = x + iy$ and $a = \alpha + i\beta$ with the respective conjugates $\overline{z} = x - iy$ and

$\overline{a} = \alpha - i\beta$ and $|a|^2 = \alpha^2 + \beta^2$. In terms of the coordinates α, β (3.13) has the form

$$f_{\alpha,\beta}(x,y) = \frac{1-\alpha^2-\beta^2}{|1-(\alpha-i\beta)(x+iy)|^2}. \tag{3.14}$$

First we will simplify the denominator $|1-(\alpha-i\beta)(x+iy)|^2$ in the following way:

$$|1-(\alpha-i\beta)(x+iy)|^2 = |1-(\alpha x+\beta y) - i(\alpha y - \beta x)|^2$$

$$= [1-(\alpha x+\beta y)]^2 + [\alpha y-\beta x]^2 = 1-2(\alpha x+\beta y)+(\alpha x+\beta y)^2+(\alpha y-\beta x)^2$$
$$= 1-2\alpha x-2\beta y+\alpha^2x^2+2\alpha\beta xy+\beta^2y^2+\alpha^2y^2-2\alpha\beta xy+\beta^2x^2$$
$$= 1-2\alpha x-2\beta y+(\alpha^2+\beta^2)(x^2+y^2).$$

We will consider the last expression as a function $g_{\alpha,\beta}(x,y)$, i.e.

$$g_{\alpha,\beta}(x,y) = 1-2\alpha x-2\beta y+(\alpha^2+\beta^2)(x^2+y^2).$$

Thus, the function $f_{\alpha,\beta}(x,y)$ is given by

$$f_{\alpha,\beta}(x,y) = \frac{1-(\alpha^2+\beta^2)}{g_{\alpha,\beta}(x,y)}.$$

The first partial derivatives are given by

$$\begin{aligned}\frac{\partial f_{\alpha,\beta}(x,y)}{\partial x} &= \frac{\partial\left(\frac{1-(\alpha^2+\beta^2)}{g_{\alpha,\beta}(x,y)}\right)}{\partial x} = \left[(\alpha^2+\beta^2)-1\right]\frac{\frac{g_{\alpha,\beta}(x,y)}{\partial x}}{[g_{\alpha,\beta}(x,y)]^2}\\ &= \left[1-(\alpha^2+\beta^2)\right]\frac{2\alpha-2x(\alpha^2+\beta^2)}{[g_{\alpha,\beta}(x,y)]^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f_{\alpha,\beta}(x,y)}{\partial y} &= \frac{\partial\left(\frac{1-(\alpha^2+\beta^2)}{g_{\alpha,\beta}(x,y)}\right)}{\partial y} = \left[(\alpha^2+\beta^2)-1\right]\frac{\frac{g_{\alpha,\beta}(x,y)}{\partial y}}{[g_{\alpha,\beta}(x,y)]^2}\\ &= \left[1-(\alpha^2+\beta^2)\right]\frac{2\beta-2y(\alpha^2+\beta^2)}{[g_{\alpha,\beta}(x,y)]^2}.\end{aligned}$$

Since we are interested in obtaining the maximum for (x,y) belonging to the circle $x^2+y^2=1$ we use the method of Lagrange multipliers, i.e., we solve

$$\begin{array}{ll}\max & f_{\alpha,\beta}(x,y)\\ s.t. & x^2+y^2=1.\end{array} \tag{3.15}$$

Thus, using (3.15) we get

$$\begin{cases} \left[1-(\alpha^2+\beta^2)\right](2\alpha-2x(\alpha^2+\beta^2)) = 2\lambda x\,[g_{\alpha,\beta}(x,y)]^2 \\ \left[1-(\alpha^2+\beta^2)\right](2\beta-2y(\alpha^2+\beta^2)) = 2\lambda y\,[g_{\alpha,\beta}(x,y)]^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left[1-(\alpha^2+\beta^2)\right](\alpha-x(\alpha^2+\beta^2)) = \lambda x\,[g_{\alpha,\beta}(x,y)]^2 \\ \left[1-(\alpha^2+\beta^2)\right](\beta-y(\alpha^2+\beta^2)) = \lambda y\,[g_{\alpha,\beta}(x,y)]^2 \end{cases}$$

$$\Leftrightarrow \frac{\alpha-x(\alpha^2+\beta^2)}{x} = \frac{\beta-y(\alpha^2+\beta^2)}{y} \Leftrightarrow \frac{\alpha}{x} = \frac{\beta}{y} \Rightarrow \begin{cases} x = \dfrac{\alpha}{\beta}y \\ y = \dfrac{\beta}{\alpha}x. \end{cases}$$

From the constraint in (3.15), we have

$$x^2\left[1+\left(\frac{\beta}{\alpha}\right)^2\right] = 1 \Rightarrow x = \pm\left[1+\left(\frac{\beta}{\alpha}\right)^2\right]^{-\frac{1}{2}} \text{ and } y = \pm\frac{\beta}{\alpha}\left[1+\left(\frac{\beta}{\alpha}\right)^2\right]^{-\frac{1}{2}}.$$

This gives us the critical values (x^*, y^*) as

$$x^* = \pm\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}} \text{ and } y^* = \pm\frac{\beta}{\alpha}\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}}$$

Replacing the last values in (3.14), we obtain

$$f_{\alpha,\beta}(x^*,y^*) = \frac{1-(\alpha^2+\beta^2)}{|1-(\alpha-i\beta)(x+iy)|^2} = \frac{1-(\alpha^2+\beta^2)}{\left|1-(\alpha-i\beta)\left(\pm\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}} \pm i\frac{\beta}{\alpha}\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}}\right)\right|^2}$$

$$= \frac{1-(\alpha^2+\beta^2)}{\left|1-(\alpha-i\beta)\left(\pm\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}} \pm i\frac{\beta}{\alpha}\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}}\right)\right|^2}.$$

We need to take a look at the denominator

$$\left|1-(\alpha-i\beta)\left(\pm\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}} \pm i\frac{\beta}{\alpha}\frac{|\alpha|}{\sqrt{\alpha^2+\beta^2}}\right)\right|^2 = \left|1-\frac{\alpha-i\beta}{\sqrt{\alpha^2+\beta^2}}\left(\pm|\alpha| \pm i\frac{|\alpha|}{\alpha}\beta\right)\right|^2$$

$$= \left|1-\frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\pm(|\alpha|\,\alpha - i\,|\alpha|\,\beta) \pm (i\,|\alpha|\,\beta - \frac{|\alpha|}{\alpha}\beta)\right)\right|^2$$

$$= \left|1-\frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\pm(|\alpha|\,\alpha - \frac{|\alpha|}{\alpha}\beta^2) \pm i(|\alpha|\,\beta - |\alpha|\,\beta)\right)\right|^2$$

$$= \left|1-\frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\pm(|\alpha|\,\alpha - \frac{|\alpha|}{\alpha}\beta^2)\right)\right|^2.$$

The value $|\alpha|/\alpha = \text{sgn}\alpha$ is 1 if $\alpha > 0$ and -1 $\alpha < 0$. We also have $|\alpha|\alpha = |\alpha|^2 \text{sgn}\alpha = \alpha^2\text{sgn}\alpha$. Thus, with a certain abuse of notation we define the functions

$$\zeta(\alpha,\beta) := \begin{cases} \left|1 - \frac{1}{\sqrt{\alpha^2+\beta^2}}\left(-\beta^2 \pm \alpha^2\right)\right|^2, & \alpha > 0, \\ \left|1 - \frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\mp\alpha^2 + \beta^2\right)\right|^2, & \alpha < 0 \end{cases}$$

which is equivalent to consider the functions

$$\zeta(\alpha,\beta) = \left|1 \pm \frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\beta^2 - \alpha^2\right)\right|^2$$

since otherwise we have just the case $\left|1 \pm \sqrt{\alpha^2+\beta^2}\right|$ and, therefore, $f_{\alpha,\beta} = \left|1 \pm \sqrt{\alpha^2+\beta^2}\right| = 1 \pm |a_n|$ or 2 as an upper bound. Thus, from (3.14) we get

$$f(\alpha,\beta) = f_{\alpha,\beta}(\alpha,\beta) = \left|\frac{1-(\alpha^2+\beta^2)}{\zeta(\alpha,\beta)}\right| = \left|\frac{\kappa(\alpha,\beta)}{\zeta(\alpha,\beta)}\right|,$$

with $\kappa(\alpha,\beta) = 1 - (\alpha^2+\beta^2)$.

Since under our conditions $\kappa \geq 0$, it is enough to minimize/maximize $g(\alpha,\beta) = \dfrac{\kappa(\alpha,\beta)}{\zeta(\alpha,\beta)}$, where g is a non-negative function, i.e., $g \geq 0$. However, as we will see in this case the maximum can be greater then 2.

Let us check if g is greater than two for some (α,β), i.e., verify the conditions

$$g(\alpha,\beta) = \frac{1-(\alpha^2+\beta^2)}{\left|1 \pm \frac{\beta^2-\alpha^2}{\sqrt{\alpha^2+\beta^2}}\right|^2} \geq 2. \tag{3.16}$$

We can prove that the inequality (3.16) is true for some (α,β) by studying the limit. Consider the function

$$g(x,y) = \frac{1-(x^2+y^2)}{\left|1 \pm \frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2}.$$

for the critical points $(x,y) = (\pm 1, 0)$ and $(x,y) = (0,\pm 1)$. For the limits we get

$$\lim_{(x,y)\to(\pm 1,0)} g_1(x,y) = \lim_{(x,y)\to(\pm 1,0)} \frac{1-(x^2+y^2)}{\left|1 + \frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2} = +\infty$$

and

$$\lim_{(x,y)\to(\pm 1,0)} g_2(x,y) = \lim_{(x,y)\to(0,\pm 1)} \frac{1-(x^2+y^2)}{\left|1 - \frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2} = +\infty.$$

This can be easily seen by changing to polar coordinates, i.e. $(x,y) = (1+\rho\cos\theta, \rho\sin\theta)$. Let us start with the first function g_1

$$\lim_{(x,y)\to(\pm1,0)} \frac{1-(x^2+y^2)}{\left|1+\frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2} = \lim_{\rho\to0} \frac{|-2\rho\cos\theta-\rho^2|}{\left|1+\frac{\rho^2(2\sin^2\theta-1)-2\rho\cos\theta-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right|^2}$$

$$\begin{aligned}
&= \lim\nolimits_{\rho\to0} \frac{|-2\cos\theta-\rho|}{\left|\rho^{-1/2}\left(1+\frac{\rho^2(2\sin^2\theta-1)-2\rho\cos\theta-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)\right|^2} \\
&= \frac{|\lim_{\rho\to0} -2\cos\theta-\rho|}{\left|\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\cos\theta+1}+\rho^2(2\sin^2\theta-1)-2\rho\cos\theta-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)\right|^2} \\
&= \frac{|\lim_{\rho\to0} -2\cos\theta-\rho|}{\left|\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\cos\theta+1}-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)+\underbrace{\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\rho^2(2\sin^2\theta-1)-2\rho\cos\theta}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)}_{=0}\right|^2} \\
&= \frac{|\lim_{\rho\to0} -2\cos\theta-\rho|}{\left|\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\cos\theta+1}-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)+\underbrace{\lim_{\rho\to0}\left(\frac{\rho^{3/2}(2\sin^2\theta-1)-2\rho^{1/2}\cos\theta}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)}_{=0}\right|^2} \\
&= \frac{|\lim_{\rho\to0} -2\cos\theta-\rho|}{\left|\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\cos\theta+1}-1}{\sqrt{\rho^2+2\rho\cos\theta+1}}\right)\right|^2} \\
&= \frac{|\lim_{\rho\to0} -2\cos\theta-\rho|}{\left|\lim_{\rho\to0}\rho^{-1/2}\left(\frac{\rho^2+2\rho\cos\theta+1-1}{\sqrt{\rho^2+2\rho\cos\theta+1}(\sqrt{\rho^2+2\rho\cos\theta+1}+1)}\right)\right|^2} \\
&= \frac{2|\cos\theta|}{\left|\underbrace{\lim_{\rho\to0}\left(\frac{\rho^{3/2}+2\rho^{1/2}\cos\theta}{\sqrt{\rho^2+2\rho\cos\theta+1}\left(\sqrt{\rho^2+2\rho\cos\theta+1}+1\right)}\right)}_{=0}\right|^2} = \infty.
\end{aligned}$$

In the same way, for the function g_2 we get

$$\lim_{(x,y)\to(0,\pm1)} \frac{1-(x^2+y^2)}{\left|1-\frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2} = \lim_{(x,y)\to(0,\pm1)} \frac{1-(x^2+y^2)}{\left|1+\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right|^2} = \lim_{\rho\to0} \frac{|-2\rho\sin\theta-\rho^2|}{\left|1+\frac{\rho^2(2\cos^2\theta-1)-2\rho\sin\theta-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right|^2}$$

$$= \lim_{\rho\to 0} \frac{\rho|(-2\sin\theta-\rho)|}{\rho\left|\rho^{-1/2}\left(1+\frac{\rho^2(2\cos^2\theta-1)-2\rho\sin\theta-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)\right|^2} = \lim_{\rho\to 0} \frac{|-2\sin\theta-\rho|}{\left|\rho^{-1/2}\left(1+\frac{\rho^2(2\cos^2\theta-1)-2\rho\sin\theta-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)\right|^2}$$

$$= \frac{|\lim_{\rho\to 0} -2\sin\theta-\rho|}{\left|\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\sin\theta+1}+\rho^2(2\cos^2\theta-1)-2\rho\sin\theta-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)\right|^2}$$

$$= \frac{|\lim_{\rho\to 0} -2\sin\theta-\rho|}{\left|\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\sin\theta+1}-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)+\underbrace{\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\rho^2(2\cos^2\theta-1)-2\rho\sin\theta}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)}_{=0}\right|^2}$$

$$= \frac{|\lim_{\rho\to 0} -2\sin\theta-\rho|}{\left|\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\sin\theta+1}-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)+\underbrace{\lim_{\rho\to 0}\left(\frac{\rho^{3/2}(2\cos^2\theta-1)-2\rho^{1/2}\sin\theta}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)}_{=0}\right|^2}$$

$$= \frac{|\lim_{\rho\to 0} -2\sin\theta-\rho|}{\left|\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\sqrt{\rho^2+2\rho\sin\theta+1}-1}{\sqrt{\rho^2+2\rho\sin\theta+1}}\right)\right|^2}$$

$$= \frac{|\lim_{\rho\to 0} -2\sin\theta-\rho|}{\left|\lim_{\rho\to 0}\rho^{-1/2}\left(\frac{\rho^2+2\rho\sin\theta+1-1}{\sqrt{\rho^2+2\rho\sin\theta+1}(\sqrt{\rho^2+2\rho\sin\theta+1}+1)}\right)\right|^2}$$

$$= \frac{|-2\sin\theta|}{\left|\underbrace{\lim_{\rho\to 0}\left(\frac{\rho^{3/2}+2\rho^{1/2}\sin\theta}{\sqrt{\rho^2+2\rho\sin\theta+1}\left(\sqrt{\rho^2+2\rho\sin\theta+1}+1\right)}\right)}_{=0}\right|^2} = \infty.$$

Since we need a finite maximum value at the two regions we have to calculate the limit of g_1 at the point $(x,y) = (\pm(1-\epsilon),0)$. Thus, we get

$$\lim_{(x,y)\to(1-\epsilon,0)} \frac{1-(x^2+y^2)}{\left|1+\frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2} = \lim_{x\to 1-\epsilon}\left(\lim_{y\to 0}\frac{1-(x^2+y^2)}{\left|1+\frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2}\right) = \lim_{x\to 1-\epsilon}\left(\frac{1-x^2}{\left|1-\frac{x^2}{\sqrt{x^2}}\right|^2}\right)$$

$$= \lim_{x\to 1-\epsilon}\left(\frac{1-x^2}{|1-x|^2}\right) = \lim_{x\to 1-\epsilon}\left(\frac{1-x^2}{|1-x|^2}\right) = \lim_{x\to 1-\epsilon}\left(\frac{(1-x)(1+x)}{(1-x)^2}\right)$$

$$= \lim_{x\to 1-\epsilon}\left(\frac{1+x}{1-x}\right) = \frac{2}{\epsilon} - 1.$$

In the same way we have

$$\lim_{y\to 0}\left(\lim_{x\to 1-\epsilon}\frac{1-(x^2+y^2)}{\left|1+\frac{y^2-x^2}{\sqrt{x^2+y^2}}\right|^2}\right) = \frac{2}{\epsilon} - 1.$$

On the other hand, for the points $(x, y) = (0, \pm(1-\epsilon))$, with a small ϵ, we obtain same limit for the g_2 function.

Thus, the maximum value of the function, under our conditions will be

$$\max_{a_n\in\mathbb{D}}\max_{z_A\in\mathbb{T}}\frac{1-|a_n|^2}{|1-\overline{a}_n z|^2} \leq \frac{2}{\epsilon} - 1. \tag{3.17}$$

□

Lemma 3.2.6. *If $N \geq |T|$ then $\mathcal{F}_{TX}$ is injective almost surely.*

Proof. We need to show now that with high probability there exists a vector P with the properties assumed in Lemma 3.2.4. The restriction operator $R_T : \ell_2(D) \to \ell_2(T)$ is given by $R_T c_k = c_k$ for $k \in T$. Its adjoint $R_T^* = E_T : \ell_2(T) \to \ell_2(D)$ is the operator that extends a vector outside T by zero, i.e., $(E_T d)_k = d_k$ for $k \in T$ and $(E_T d)_k = 0$ otherwise.

Now assume for the moment that $\mathcal{F}_{TX}^*\mathcal{F}_{TX} : \ell_2(T) \to \ell_2(T)$ is invertible. In this case we define P explicitly by

$$P := \mathcal{F}_X^*\mathcal{F}_{TX}(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}(c),$$

where as before $T := \mathrm{supp}c$. Then clearly P has property (i) and property (iii) in Lemma 3.2.4 with

$$\lambda := \mathcal{F}_{TX}(\mathcal{F}_{TX}^*\mathcal{F}_{TX})^{-1}R_T\mathrm{sgn}(c) \in \ell_2(X).$$

We are left with proving that P has property (ii) of Lemma 3.2.4 with high probability.

To this end we introduce the auxiliary operators

$$H := \ell_2(T) \to \ell_2(D), \qquad H := DE_T - \mathcal{F}_X^*\mathcal{F}_{TX}$$

and

$$H_0 := \ell_2(T) \to \ell_2(T), \qquad H_0 := R_T H = DI_T - \mathcal{F}_{TX}^*\mathcal{F}_{TX},$$

where I_T denotes the identity on $l^2(T)$. The diagonal matrix D can be written as

$$D_{mm} = \sum_{j=1}^{N}\frac{\sqrt{1-|a_m|^2}}{1-a_m e^{-ix_j}}\prod_{\ell=0}^{m-1}\frac{e^{-ix_j}-\overline{a_\ell}}{1-a_\ell e^{-ix_j}}\frac{\sqrt{1-|a_m|^2}}{1-\overline{a_m}e^{ix_j}}\prod_{\ell=0}^{m-1}\frac{e^{ix_j}-a_\ell}{1-\overline{a_\ell}e^{ix_j}}.$$

Obviously, H_0 is self-adjoint, and

$$H = [h_{mn}] := D_{mn}\delta_{mn} - \sum_{j=1}^{N} \frac{\sqrt{1-|a_m|^2}}{1-a_m e^{-ix_j}} \prod_{\ell=0}^{m-1} \frac{e^{-ix_j}-\overline{a_\ell}}{1-a_\ell e^{-ix_j}} \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}e^{ix_j}} \prod_{k=0}^{n-1} \frac{e^{ix_j}-a_k}{1-\overline{a_k}e^{ix_j}} \tag{3.18}$$

acts on a vector as

$$(Hc)_m = - \sum_{n=1,n\neq m}^{d} \sum_{j=1}^{N} \frac{\sqrt{1-|a_m|^2}}{1-a_m e^{-ix_j}} \prod_{\ell=0}^{m-1} \frac{e^{-ix_j}-\overline{a_\ell}}{1-a_\ell e^{-ix_j}} \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}e^{ix_j}} \prod_{k=0}^{n-1} \frac{e^{ix_j}-a_k}{1-\overline{a_k}e^{ix_j}} c_n. \tag{3.19}$$

Now we can write

$$P = (DE_T - H)(DI_T - H_0)^{-1} R_T \mathrm{sgn}(c).$$

As we are interested in property (ii) in Lemma 3.2.4 we consider only values of P on $T^c = D \setminus T$. Since $R_{T^c}E_T = 0$ we have

$$P_k = -D^{-1} R_{T^c} H \Big(I_T - D^{-1}H_0 \Big)^{-1} R_T \mathrm{sgn}(c) \qquad \text{for all} \quad k \in T^c.$$

Again, we can study the term $\Big(I_T - D^{-1}H_0 \Big)^{-1}$ via the von Neumann series (see, for instance [54]) of
$\Big(I_T - (D^{-1}H_0)^n \Big)^{-1} = I_T + A_n$ with

$$A_n := \sum_{r=1}^{\infty} \Big(D^{-1}H_0 \Big)^{rn}, \qquad n \in \mathbb{N}. \tag{3.20}$$

As before, we get

$$\Big(I_T - (D^{-1}H_0) \Big)^{-1} = (I_T + A_n) \sum_{m=0}^{n-1} \big(D^{-1}H_0 \big)^m.$$

Therefore, we can write on the complement of T

$$R_{T^c}P = H(I_T + A_n)\Big(\sum_{m=0}^{n-1} \big(D^{-1}H_0 \big)^m \Big) D^{-1} R_T \mathrm{sgn}(c) = -\big(P^{(1)} + P^{(2)} \big),$$

where

$$P^{(1)} = DS_n D^{-1}\mathrm{sgn}(c) \quad \text{and} \quad P^{(2)} = HA_n R_T (I + S_{n-1}) D^{-1} \mathrm{sgn}(c),$$

with $S_n := \sum_{m=0}^{n-1} \big(D^{-1} H R_T \big)^m$.

Again, as our goal is to estimate $\mathbb{P}(\sup_{k\in T^c}|P_k| \geq 1)$ we consider $a_1, a_2 > 0$ to be numbers satisfying $a_1 + a_2 = 1$ and we have

$$\mathbb{P}\big(\sup_{k\in T^c}|P_k| \geq 1\big) \leq \mathbb{P}\big(\big\{\sup_{k\in T^c}|P_k^{(1)}| \geq a_1\big\} \cup \big\{\sup_{k\in T^c}|P_k^{(2)}| \geq a_2\big\}\big). \tag{3.21}$$

This leads to the estimates

$$\begin{aligned} \mathbb{P}\big(|P_k^{(1)}| \geq a_1\big) &= \mathbb{P}\left(\left|\left(DS_nD^{-1}\mathrm{sgn}(c)\right)_k\right| \geq a_1\right) \\ \leq \mathbb{P}(E_k) &:= \mathbb{P}\bigg(\sum_{m=1}^{n} |(D(D^{-1}HR_T)^m D^{-1}\mathrm{sgn}(c))_k| \geq a_1\bigg). \end{aligned}$$

and

$$\begin{aligned} \sup_{k\in T^c}|P_k^{(2)}| &\leq \left\|P^{(2)}\right\|_\infty \\ &\leq \left\|HA_nD^{-1}\right\|_{\ell^\infty(T)\to\ell^\infty(D)} \left(1 + \|R_TDS_{n-1}D^{-1}\mathrm{sgn}(c)\|_{\ell^\infty(T)}\right) \end{aligned} \tag{3.22}$$

where $\ell^\infty(D)$ denote the space of sequences indexed by D.
For the term $\|R_TDS_{n-1}D^{-1}\mathrm{sgn}(c)\|_{\ell^\infty}(T)$ similarly as in (3.22) we get

$$\begin{aligned} \mathbb{P}\big(|(DS_{n-1}D^{-1}\mathrm{sgn}(c))_k| \geq a_1\big) \leq \mathbb{P}(E_k) &= \mathbb{P}\bigg(\sum_{m=1}^{n} |(D(D^{-1}HR_T)^m D^{-1}\mathrm{sgn}(c))_k| \geq a_1\bigg) \\ &= \mathbb{P}\bigg(\sum_{m=1}^{n} |((HD^{-1}R_T)^m D^{-1}\mathrm{sgn}(c))_k| \geq a_1\bigg). \end{aligned}$$

Now, we know as well that

$$\left\|HA_nD^{-1}\right\|_\infty \leq \left\|HD^{-1}DA_nD^{-1}\right\|_\infty \leq \left\|HD^{-1}\right\|_\infty \left\|DA_nD^{-1}\right\|_\infty. \tag{3.23}$$

with $DA_nD^{-1} = \sum_{r=1}^{\infty} D\left(D^{-1}H_0\right)^{rn} D^{-1}$.
For the estimation of the term $\left\|DA_nD^{-1}\right\|_\infty$ we can use the Frobenius norm (see, for instance [45] and [59]). Note that the trace of product of the matrix with another diagonal matrix D with same dimension is commutative, i.e., $Tr(AD) = Tr(DA)$. Therefore, in our case $\left\|DA_nD^{-1}\right\|_F^2 = Tr(A_nA_n^*)$. Thus, for now, we suppose that

$$\left\|D\left(D^{-1}H_0\right)^n D^{-1}\right\|_F \leq \kappa < 1. \tag{3.24}$$

From the definition (3.20) of A_n, it follows that

$$\begin{aligned} \left\|DA_nD^{-1}\right\|_F &= \left\|D\sum_{r=1}^{\infty}\left(D^{-1}H_0\right)^{rn} D^{-1}\right\|_F \leq \sum_{r=1}^{\infty} \left\|D(D^{-1}H_0)^n D^{-1}\right\|_F^r \\ &\leq \sum_{r=1}^{\infty} \kappa^r = \frac{\kappa}{1-\kappa}. \end{aligned} \tag{3.25}$$

For the term $\left\|HD^{-1}\right\|_\infty$, we have to remind us that the matrix H is given by

$$H = [h_{\ell k}] := D_{\ell k} - \sum_{j=1}^{N} \frac{\sqrt{1-|a_\ell|^2}}{1-a_\ell e^{-ix_j}} \prod_{s=0}^{\ell-1} \frac{e^{ix_j}-a_s}{1-\overline{a_s}e^{ix_j}} \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}e^{ix_j}} \prod_{m=0}^{k-1} \frac{e^{-ix_j}-\overline{a_m}}{1-a_m e^{-ix_j}},$$

such that by (3.18) and $\left|\prod_{s=0}^{\ell-1} \frac{e^{ix}-a_s}{1-\overline{a_s}e^{ix}}\right| = 1$ and $\left|\prod_{m=0}^{k-1} \frac{e^{-ix}-\overline{a_m}}{1-a_m e^{-ix}}\right| = 1$ we obtain

$$\begin{aligned}
\left\|HD^{-1}\right\|_\infty &= \left\| \left(\sum_{j=1}^{N} \frac{\sqrt{1-|a_\ell|^2}}{1-a_\ell e^{-ix_j}} \prod_{s=0}^{\ell-1} \frac{e^{ix_j}-a_s}{1-\overline{a_s}e^{ix_j}} \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}e^{ix_j}} \prod_{m=0}^{k-1} \frac{e^{-ix_j}-\overline{a_m}}{1-a_m e^{-ix_j}} D_{kk}^{-1} \right)_{\ell k} \right\|_\infty \\
&= \sup_\ell \sum_{\ell\neq k\in T} \left| \sum_{j=1}^{N} \frac{\sqrt{1-|a_\ell|^2}}{1-a_\ell e^{-ix_j}} \prod_{s=0}^{\ell-1} \frac{e^{ix_j}-a_s}{1-\overline{a_s}e^{ix_j}} \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}e^{ix_j}} \prod_{m=0}^{k-1} \frac{e^{-ix_j}-\overline{a_m}}{1-a_m e^{-ix_j}} D_{kk}^{-1} \right|
\end{aligned}$$

$$\leq \sup_\ell \sum_{\ell\neq k\in T} \left(\sum_{j=1}^{N} \left| \frac{\sqrt{1-|a_\ell|^2}}{1-\overline{a_\ell}e^{i\cdot x_j}} \right| \left| \prod_{s=0}^{\ell-1} \frac{e^{ix_j}-a_s}{1-\overline{a_s}e^{ix_j}} \right| \left| \frac{\sqrt{1-|a_k|^2}}{1-a_k e^{-i\cdot x_j}} \right| \left| \prod_{m=0}^{k-1} \frac{e^{-ix_j}-\overline{a_m}}{1-a_m e^{-ix_j}} D_{kk}^{-1} \right| \right)$$

$$\begin{aligned}
&\leq \sup_\ell \sum_{\ell\neq k\in T} \sum_{j=1}^{N} \left| \frac{\sqrt{1-|a_\ell|^2}}{1-\overline{a_\ell}e^{i\cdot x_j}} \frac{\sqrt{1-|a_k|^2}}{1-a_k e^{-i\cdot x_j}} D_{kk}^{-1} \right| \\
&\leq \sup_\ell \sum_{\ell\neq k\in T} \sum_{j=1}^{N} \left| \frac{\sqrt{1-|a_\ell|^2}}{1-\overline{a_\ell}e^{i\cdot x_j}} \right| \left| \frac{\sqrt{1-|a_k|^2}}{1-a_k e^{-i\cdot x_j}} \right| \left|D_{kk}^{-1}\right|. \qquad (3.26)
\end{aligned}$$

Using the estimate from Lemma 3.2.5 in last expression, we can further estimate

$$\begin{aligned}
\sup_\ell \sum_{\ell\neq k\in T} \sum_{j=1}^{N} \sqrt{\frac{2}{\epsilon}-1}\sqrt{\frac{2}{\epsilon}-1} \left|D_{kk}^{-1}\right| &\leq \sup_\ell \sum_{\ell\neq k\in T} N\left(\frac{2}{\epsilon}-1\right)\left|D_{kk}^{-1}\right| \\
= N\left(\frac{2}{\epsilon}-1\right) \sup_\ell \sum_{\ell\neq k\in T} \left|D_{kk}^{-1}\right| &\leq N\frac{2}{\epsilon} \sup_\ell \sum_{\ell\neq k\in T} \left|D_{kk}^{-1}\right|.
\end{aligned}$$

We recall that

$$\begin{aligned}
|D_{kk}| &= \sum_{j=1}^{N} \frac{1-|a_\ell|^2}{\left|1-\overline{a_\ell}e^{i\cdot x_j}\right|^2} \geq N\cdot \min_j \frac{1-|a_\ell|^2}{\left|1-\overline{a_\ell}e^{i\cdot x_j}\right|^2} \geq N\frac{1-|a_\ell|^2}{(1+|a_\ell|)^2} \\
&= N\frac{1-|a_\ell|}{1+|a_\ell|} \geq N\frac{\epsilon}{2}. \qquad (3.27)
\end{aligned}$$

Then, taking in account (3.27), expression (3.26) be further simplified

$$\sup_\ell \sum_{\ell\neq k\in T} \sum_{j=1}^{N} \left| \frac{\sqrt{1-|a_\ell|^2}}{1-\overline{a_\ell}e^{i\cdot x_j}} \right| \left| \frac{\sqrt{1-|a_k|^2}}{1-a_k e^{-i\cdot x_j}} \right| \left|D_{kk}^{-1}\right| \leq N\frac{2}{\epsilon} \sup_\ell \sum_{\ell\neq k\in T} \left|D_{kk}^{-1}\right|$$

$$\frac{N\epsilon}{2}\frac{2}{N\epsilon}(|T|-1) \leq |T|. \tag{3.28}$$

Now, from (3.23), (3.25) and (3.28), we estimate

$$\begin{aligned}\left\|HA_nD^{-1}\right\|_\infty &\leq \left\|HD^{-1}DA_nD^{-1}\right\|_\infty \leq \left\|HD^{-1}\right\|_\infty \left\|DA_nD^{-1}\right\|_\infty \\ &\leq |T|\,\frac{\kappa}{1-\kappa}. \end{aligned} \tag{3.29}$$

Thus, from (3.24) and $\|S_{n-1}\mathrm{sgn}(c)\|_\infty < a_1$ it follows

$$\sup_{k\in T^c}|P_k^{(2)}| \quad \leq \quad (1+a_1)\frac{\kappa}{1-\kappa}\,|T|^{\frac{3}{2}}\,. \tag{3.30}$$

Therefore, taking in account the estimate (3.27) from (3.30) and if

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}\,|T|^{-\frac{3}{2}} \tag{3.31}$$

then $\sup_{k\in T^c}|P_k^{(2)}| \geq a_2$ as intended.

Also it follows from (3.31) that $\kappa < 1$ and $|T| \geq 1$ (note that if $T = \emptyset$ then $c = 0$) and ℓ_1-minimization will clearly recover f.

Let us look again at the cases of Theorem 3.2.1 where the variable is deterministic and of Theorem 3.2.3 where $|T|$ is a random variable.

1. In the first case, where $|T|$ is random by Theorem 3.2.3 if

$$|T| \leq (\alpha+1)\mathbb{E}\,|T| \tag{3.32}$$

with $\alpha > 0$ and by assuming

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1}((\alpha+1)\mathbb{E}\,|T|)^{-\frac{3}{2}} \tag{3.33}$$

we see that (3.31) is satisfied and we have

$$\sup_{t\in T^c}\left|P_k^{(2)}\right| \leq a_2.$$

From (3.21) we can again estimate

$$\begin{aligned}&\mathbb{P}\big(\sup_{k\in T^c}|P_k| \geq 1\big) \leq \mathbb{P}\Bigg(\bigcup_{k\in T^c}\left\{\left|P_k^{(1)}\right| \geq a_1\right\} \cup \left\{\,\|R_T\mathrm{sgn}(c)\|_{\ell^\infty(T)} \geq a_1\right\} \\ \cup\;\; &\left\{\left\|(D^{-1}H_0)^n\right\|_F \geq \kappa\right\} \cup \left\{\,|T| \geq (\alpha+1)\mathbb{E}\,|T|\right. \\ \leq\;\; &\mathbb{P}\Bigg(\bigcup_{k\in D}E_k \cup \left\{\left\|(D^{-1}H_0)^n\right\|_F \geq \kappa\right\} \cup \left\{\,|T| \geq (\alpha+1)\mathbb{E}\,|T|\,\right\}\Bigg) \\ \leq\;\; &\sum_{k\in D}\mathbb{P}(E_k) + \mathbb{P}\big(\left\|(D^{-1}H_0)^n\right\|_F \geq \kappa\big) + \mathbb{P}\big(\,|T| \geq (\alpha+1)\mathbb{E}\,|T|\,. \end{aligned} \tag{3.34}$$

2. In case of Theorem 3.2.1 we do not assume that $|T|$ is a random variable. From condition (3.31), we obtain

$$\mathbb{P}\big(\sup_{k\in T^c}|P_k| \geq 1\big) \leq \sum_{k\in D} \mathbb{P}(E_k) + \mathbb{P}\big(\left\|(D^{-1}H_0)^n\right\|_F \geq \kappa\big). \tag{3.35}$$

Thus, as in the previous cases we need to estimate $\mathbb{P}(E_k)$ and $\mathbb{P}\big(\left\|(D^{-1}H_0)^n\right\|_F \geq \kappa\big)$. The proof until this point is basically the same as in the previous chapters.

□

3.2.3 Analysis of powers of G_0

As in the other cases we need to estimate the powers of the random matrix $G_0 = D^{-1}H_0$ in Frobenius norm. We will only calculate the expectation value $\mathbb{E}_X$. We leave the computation of the full expectation $\mathbb{E} = \mathbb{E}_T\mathbb{E}_X$ for the situation of Theorem 3.2.1 where it is only affected by the set partitions. The principal result is the following lemma.

Lemma 3.2.7. *It holds*

$$\mathbb{E}_X[\|G_0^n\|_F^2] \leq \sum_{t=1}^{\min\{n,N\}} \frac{N!}{(N-t)!} \sum_{\mathcal{A}\in P(2n,t)} \mathcal{C}_D\, \mathcal{C}(\mathcal{A},T),$$

with

$$\mathcal{C}_D := \left(\prod_{s=1}^{2n} \sum_{l_s=1}^{N} \left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2 \right)^{-1} \tag{3.36}$$

and

$$\mathcal{C}(\mathcal{A},T) := \sum_{k_1,k_2,\ldots,k_{2n}\in T, k_r\neq k_{r+1}} \prod_{A\in\mathcal{A}} \left(\frac{2}{\epsilon} - 1\right)^t \tag{3.37}$$

for $|\pm 1 - a_n| = \epsilon$ *and* $|\pm i - a_n| = \epsilon$ *with a small* ϵ.

Proof. As first step we have to remark that it will be a quite hard problem to calculate the expectation value with respect to a Blaschke product and, therefore, we are more interested to get an estimate. Let us start by taking a closer look at the self-adjoint matrix G_0 . We need to estimate $\|G_0^n\|_F^2 = \mathrm{Tr}(G_0^{2n})$.

Since we have

$$\begin{aligned} \mathcal{F}_X &:= \left[\mathcal{B}_k(e^{ix_j})\right]_{j=1,\ldots,N,k=1,\ldots,d} \\ &= \left[\frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}e^{ix_j}} \prod_{t=0}^{k-1} \frac{e^{ix_j}-a_t}{1-\overline{a_t}e^{ix_j}} \right]_{j=1,\ldots,N,k=1,\ldots,d}. \end{aligned} \tag{3.38}$$

we know that the diagonal matrix D is given by

$$D := \left[\sum_{j=1}^{N} \left|\mathcal{B}_k(e^{ix_j})\right|^2 \delta_{k\ell}\right]_{k,\ell\in T}$$

$$= \begin{bmatrix} \sum_{j=1}^{N} \left|\mathcal{B}_{k_1}(e^{ix_j})\right|^2 & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{N} \left|\mathcal{B}_{k_2}(e^{ix_j})\right|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{N} \left|\mathcal{B}_{k_{|T|}}(e^{ix_j})\right|^2 \end{bmatrix}.$$

Thus the inverse matrix of D can be written as

$$D^{-1} := \left[\sum_{j=1}^{N} \left|\mathcal{B}_k(e^{ix_j})\right|^2 \delta_{k\ell}\right]_{k,\ell\in T}^{-1}$$

$$= \begin{bmatrix} 1/\sum_{j=1}^{N} \left|\mathcal{B}_{k_1}(e^{ix_j})\right|^2 & 0 & \cdots & 0 \\ 0 & 1/\sum_{j=1}^{N} \left|\mathcal{B}_{k_2}(e^{ix_j})\right|^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/\sum_{j=1}^{N} \left|\mathcal{B}_{k_{|T|}}(e^{ix_j})\right|^2 \end{bmatrix}$$

and, additionally, we have the matrix H_0

$$H_0 := \left[(1-\delta_{lk}) \sum_{j=1}^{N} \overline{\mathcal{B}_l(e^{ix_j})}\mathcal{B}_k(e^{ix_j})\right]_{k,l\in T}$$

$$= \begin{bmatrix} 0 & \sum_{j=1}^{N} \overline{\mathcal{B}_{l_1}(e^{ix_j})}\mathcal{B}_{k_2}(e^{ix_j}) & \cdots & \sum_{j=1}^{N} \overline{\mathcal{B}_{l_1}(e^{ix_j})}\mathcal{B}_{k_{|T|}}(e^{ix_j}) \\ \sum_{j=1}^{N} \overline{\mathcal{B}_{l_2}(e^{ix_j})}\mathcal{B}_{k_1}(e^{ix_j}) & 0 & \cdots & \sum_{j=1}^{N} \overline{\mathcal{B}_{l_2}(e^{ix_j})}\mathcal{B}_{k_{|T|}}(e^{ix_j}) \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{N} \overline{\mathcal{B}_{l_{|T|}}(e^{ix_j})}\mathcal{B}_{k_1}(e^{ix_j}) & \sum_{j=1}^{N} \overline{\mathcal{B}_{l_{|T|}}(e^{ix_j})}\mathcal{B}_{k_2}(e^{ix_j}) & \cdots & 0 \end{bmatrix}.$$

Consider now the entrance (l,k) of the matrices $F_0 = H_0 D^{-1}$ and $G_0 = D^{-1}H_0$. Here we get

$$F_0(l,k) := (H_0 D^{-1})_{lk} = (1-\delta_{lk}) \frac{\mathcal{B}_l \bullet \mathcal{B}_k}{\|\mathcal{B}_k\|^2}, \tag{3.39}$$

and the norm

$$G_0(l,k) := (D^{-1}H_0)_{lk} = (1-\delta_{lk})\frac{\mathcal{B}_l \bullet \mathcal{B}_k}{\|\mathcal{B}_l\|^2}, \tag{3.40}$$

where we used the scalar product

$$\mathcal{B}_l \bullet \mathcal{B}_k = \sum_{j=1}^{N} \overline{\mathcal{B}_l(e^{ix_j})}\mathcal{B}_k(e^{ix_j})$$

and the norm

$$\|\mathcal{B}_k\|^2 = \sum_{j=1}^{N} \left|\mathcal{B}_k(e^{ix_j})\right|^2 \ \text{ and } \ \|\mathcal{B}_l\|^2 = \sum_{j=1}^{N} \left|\mathcal{B}_l(e^{ix_j})\right|^2.$$

The objective is to estimate the square of Frobenius norm of (3.40), i.e., $\left\|\left(D_T^{-1}H_0\right)^n\right\|_F^2 = \operatorname{Tr}\left[\left(D_T^{-1}H_0\right)^{2n}\right] = \operatorname{Tr}\left[G_0^{2n}\right]$.

From (3.40) we obtain the first power of G_0. So, we need to calculate $G_0^{2n}(k_1, k_{2n+1})$. Taking into account that for the calculation of the trace we have $k_1 = k_{2n+1}$ for all following powers, we get

$$\begin{aligned} G_0^2(k_1,k_1) &= \sum_{s\in T,\, s\neq k_1} G_0(k_1,s)G_0(s,k_1) = \sum_{s\in T,\, s\neq k_1} (1-\delta_{k_1 s})^2 \frac{\mathcal{B}_{k_1}\bullet \mathcal{B}_s}{\|\mathcal{B}_{k_1}\|^2}\frac{\overline{\mathcal{B}_{k_1}\bullet \mathcal{B}_s}}{\|\mathcal{B}_{k_s}\|^2} \\ &= \sum_{s\in T,\, s\neq k_1} (1-\delta_{k_1 s})^2 \frac{\mathcal{B}_{k_1}\bullet \mathcal{B}_s}{\|\mathcal{B}_{k_1}\|^2}\frac{\mathcal{B}_s\bullet \mathcal{B}_{k_1}}{\|\mathcal{B}_{k_s}\|^2}. \end{aligned}$$

Note that if $s = k_2$, we will get a meaningless zero matrix. Thus, we obtain

$$\begin{aligned} G_0^2(k_1,k_1) &= \sum_{k_2\in T,\, k_2\neq k_1} \frac{\mathcal{B}_{k_1}\bullet \mathcal{B}_{k_2}}{\|\mathcal{B}_{k_1}\|^2}\frac{\mathcal{B}_{k_2}\bullet \mathcal{B}_{k_1}}{\|\mathcal{B}_{k_2}\|^2} \\ &= \sum_{k_2\in T, k_2\neq k_1}\ \sum_{j_1,j_2=1}^{N} \frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_1}(e^{ix_{j_2}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}. \end{aligned}$$

From now on to simplify the notation we will always assume $k_j \neq k_{j+1}$ without explicitly mentioning it. For the case of $n = 2$ we have

$$\begin{aligned} G_0^4(k_1,k_1) &= \sum_{k_2,k_3,k_4\in T} G_0(k_1,k_2)G_0(k_2,k_3)G_0(k_3,k_4)G_0(k_4,k_1) \\ &= \sum_{k_2,\dots,k_{2n}\in T} \frac{\mathcal{B}_{k_1}\bullet \mathcal{B}_{k_2}}{\|\mathcal{B}_{k_1}\|^2}\frac{\mathcal{B}_{k_2}\bullet \mathcal{B}_{k_3}}{\|\mathcal{B}_{k_2}\|^2}\ \frac{\mathcal{B}_{k_3}\bullet \mathcal{B}_{k_4}}{\|\mathcal{B}_{k_3}\|^2}\frac{\mathcal{B}_{k_4}\bullet \mathcal{B}_{k_1}}{\|\mathcal{B}_{k_4}\|^2} \\ &= \sum_{k_2,k_3,k_4\in T}\ \sum_{j_1,j_2,j_3,j_4=1}^{N} \frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_3}(e^{ix_{j_2}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2} \end{aligned}$$

$$\times \frac{\overline{\mathcal{B}_{k_3}(e^{ix_{j_3}})}\mathcal{B}_{k_4}(e^{ix_{j_3}})}{\sum_{l_3=1}^{N}\left|\mathcal{B}_{k_3}(e^{ix_{l_3}})\right|^2} \frac{\overline{\mathcal{B}_{k_4}(e^{ix_{j_4}})}\mathcal{B}_{k_1}(e^{ix_{j_4}})}{\sum_{l_4=1}^{N}\left|\mathcal{B}_{k_4}(e^{ix_{l_4}})\right|^2}.$$

Furthermore, the general form of $G_0^{2n}(k_1, k_1)$ is given by

$$G_0^{2n}(k_1, k_1) = \sum_{k_2,\ldots,k_{2n}\in T} G_0(k_1, k_2)G_0(k_2, k_3)\cdots G_0(k_{2n}, k_1) = \sum_{k_2,\ldots,k_{2n}\in T}\prod_{r=1}^{2n} G_0(k_r, k_{r+1}),$$

taking $k_{2n+1} = k_1$. In a more detailed form we have

$$\begin{aligned} G_0^{2n}(k_1, k_1) &= \sum_{k_2,\ldots,k_{2n}\in T} \frac{\mathcal{B}_{k_1}\bullet\mathcal{B}_{k_2}}{\|\mathcal{B}_{k_1}\|^2}\frac{\mathcal{B}_{k_2}\bullet\mathcal{B}_{k_3}}{\|\mathcal{B}_{k_2}\|^2}\frac{\mathcal{B}_{k_3}\bullet\mathcal{B}_{k_4}}{\|\mathcal{B}_{k_3}\|^2}\frac{\mathcal{B}_{k_4}\bullet\mathcal{B}_{k_5}}{\|\mathcal{B}_{k_4}\|^2}\cdots\frac{\mathcal{B}_{k_{2n-1}}\bullet\mathcal{B}_{k_{2n}}}{\left\|\mathcal{B}_{k_{2n-1}}\right\|^2}\frac{\mathcal{B}_{k_{2n}}\bullet\mathcal{B}_{k_1}}{\|\mathcal{B}_{k_{2n}}\|^2} \\ &= \sum_{k_2,\ldots,k_{2n}\in T}\sum_{j_1,\ldots,j_{2n}=1}^{N} \frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_3}(e^{ix_{j_2}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}\cdots\frac{\overline{\mathcal{B}_{k_{2n}}(e^{ix_{j_{2n}}})}\mathcal{B}_{k_1}(e^{ix_{j_{2n}}})}{\sum_{l_{2n}=1}^{N}\left|\mathcal{B}_{k_{2n}}(e^{ix_{l_{2n}}})\right|^2}. \end{aligned}$$

As in the previous cases we have to switch to a partition $\mathcal{A}$. Furthermore, if $A \in \mathcal{A}$ contains only a pair of elements $\overline{\mathcal{B}_{a_{t_r}}(z_A)}\mathcal{B}_{a_{t_{r+1}}}(z_A)$ then the term will vanish due to the condition $k_r \neq k_{r+1}$. Thus, we only need to consider partitions $\mathcal{A}$ satisfying $|A| \geq 2$ for all $A \in \mathcal{A}$, i.e., partitions in $P(2n, t)$ with $t > 1$. Furthermore, remember that the number of vectors $(\ell_{A_1}, \ldots, \ell_{A_t}) \in \{1, \ldots, N\}^t$ with different entries is exactly $N\cdots(N-t+1) = N!/(N-t)!$ if $N \geq t$ and 0 if $N \leq t$. Thus, we obtain the expectation value of the trace of the matrix G_0 as

$$\begin{aligned} \mathbb{E}_X\left[\mathrm{Tr}\left(G_0^2\right)\right] &= \mathbb{E}_X\left[\sum_{k1,k_2\in T}\sum_{j_1,j_2=1}^{N}\frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_1}(e^{ix_{j_2}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}\right] \\ &= \sum_{k1,k_2\in T}\sum_{j_1,j_2=1}^{N}\mathbb{E}_X\left[\frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_1}(e^{ix_{j_2}})}{\sum_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}\right] \\ &= \sum_{k1,k_2\in T}\sum_{j_1,j_2=1}^{N}\mathbb{E}_X\left[\frac{\overline{\mathcal{B}_{k_1}(z_{j_1})}\mathcal{B}_{k_2}(z_{j_1})\overline{\mathcal{B}_{k_2}(z_{j_2})}\mathcal{B}_{k_1}(z_{j_2})}{\sum_{l_1,l_2=1}^{N}\left|\mathcal{B}_{k_1}(z_{j_1})\right|^2\left|\mathcal{B}_{k_2}(z_{j_2})\right|^2}\right] \end{aligned}$$

$$= \sum_{k1,k_2\in T}\sum_{j_1,j_2=1}^{N}\mathbb{E}_X\left[\frac{\overline{\mathcal{B}_{k_1}(z_{j_1})}\mathcal{B}_{k_2}(z_{j_1})\overline{\mathcal{B}_{k_2}(z_{j_2})}\mathcal{B}_{k_1}(z_{j_2})}{\sum\limits_{l_1,l_2=1}^{N}|\mathcal{B}_{k_1}(z_{j_1})|^2\,|\mathcal{B}_{k_2}(z_{j_2})|^2}\right]$$

$$\mathbb{E}_X\left[\mathrm{Tr}\left(G_0^4\right)\right] = \mathbb{E}_X\left[\sum_{k_1,k_2,k_3,k_4\in T}\sum_{j_1,j_2,j_3,j_4=1}^{N}\frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum\limits_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_3}(e^{ix_{j_2}})}{\sum\limits_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}\right.$$
$$\left.\frac{\overline{\mathcal{B}_{k_3}(e^{ix_{j_3}})}\mathcal{B}_{k_4}(e^{ix_{j_3}})}{\sum\limits_{l_3=1}^{N}\left|\mathcal{B}_{k_3}(e^{ix_{l_3}})\right|^2}\frac{\overline{\mathcal{B}_{k_4}(e^{ix_{j_4}})}\mathcal{B}_{k_1}(e^{ix_{j_4}})}{\sum\limits_{l_4=1}^{N}\left|\mathcal{B}_{k_4}(e^{ix_{l_4}})\right|^2}\right]$$
$$= \sum_{k_1,k_2,k_3,k_4\in T}\sum_{j_1,j_2,j_3,j_4=1}^{N}\mathbb{E}_X\left[\frac{\overline{\mathcal{B}_{k_1}(e^{ix_{j_1}})}\mathcal{B}_{k_2}(e^{ix_{j_1}})}{\sum\limits_{l_1=1}^{N}\left|\mathcal{B}_{k_1}(e^{ix_{l_1}})\right|^2}\frac{\overline{\mathcal{B}_{k_2}(e^{ix_{j_2}})}\mathcal{B}_{k_3}(e^{ix_{j_2}})}{\sum\limits_{l_2=1}^{N}\left|\mathcal{B}_{k_2}(e^{ix_{l_2}})\right|^2}\right.$$
$$\left.\frac{\overline{\mathcal{B}_{k_3}(e^{ix_{j_3}})}\mathcal{B}_{k_4}(e^{ix_{j_3}})}{\sum\limits_{l_3=1}^{N}\left|\mathcal{B}_{k_3}(e^{ix_{l_3}})\right|^2}\frac{\overline{\mathcal{B}_{k_4}(e^{ix_{j_4}})}\mathcal{B}_{k_1}(e^{ix_{j_4}})}{\sum\limits_{l_4=1}^{N}\left|\mathcal{B}_{k_4}(e^{ix_{l_4}})\right|^2}\right]$$

and, finally, by making the now customary switch to randomly independent variables by using partitions in the general case we get

$$\begin{aligned}
\mathbb{E}_X\left[\mathrm{Tr}\left(G_0^{2n}\right)\right] &= \sum_{k_1,k_2,\ldots,k_{2n}\in T}\sum_{j_1,\ldots,j_{2n}=1}^{N}\mathbb{E}_X\left[\frac{\prod_{r=1}^{2n}\overline{\mathcal{B}_{k_r}(e^{ix_{j_r}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_r}})}{\prod_{s=1}^{2n}\sum\limits_{l_s=1}^{N}\left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2}\right]\\
&= \sum_{k_1,k_2,\ldots,k_{2n}\in T}\sum_{j_1,\ldots,j_{2n}=1}^{N}\mathbb{E}_X\left[\frac{\prod_{r=1}^{2n}\overline{\mathcal{B}_{k_r}(e^{ix_{j_r}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_r}})}{\prod_{s=1}^{2n}\sum\limits_{l_s=1}^{N}\left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2}\right]\\
&= \sum_{k_1,k_2,\ldots,k_{2n}\in T}\sum_{j_1,\ldots,j_{2n}=1}^{N}\mathbb{E}_X\left[\frac{\prod_{A\in\mathcal{A}}\prod_{r\in A}\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})}{\prod_{s=1}^{2n}\sum\limits_{l_s=1}^{N}\left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2}\right]
\end{aligned}$$

$$= \sum_{k_1,k_2,\ldots,k_{2n}\in T} \sum_{j_1,\ldots,j_{2n}=1}^{N} \prod_{A\in\mathcal{A}} \mathbb{E}_X \left[\frac{\prod_{r\in A} \overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})}{\prod_{s=1}^{2n} \sum\limits_{l_s=1}^{N} \left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2} \right]$$

$$= \sum_{k_1,k_2,\ldots,k_{2n}\in T} \sum_{j_1,\ldots,j_{2n}=1}^{N} \prod_{A\in\mathcal{A}} \left(\frac{1}{\prod\limits_{s=1}^{2n} \sum\limits_{l_s=1}^{N} \left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2} \right) \mathbb{E}_X \left[\prod_{r\in A} \overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}}) \right]. \tag{3.41}$$

The last step is possible since the terms $\sum_{l_s=1}^{N} \left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2$ are actually fixed constants.

From last expression (3.41) since x_{j_A} has the uniform distribution on $[0, 2\pi]$ we have to look for the expectation value

$$\mathbb{E}_X \left[\prod_{r\in A} \overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}}) \right].$$

For $|A| = 1$, (note that $k_r \neq k_{r+1}$) we get

$$\mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}}) \right] = \mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}}) \right]$$

$$= \frac{1}{2\pi} \int_{|z_A|=1} \frac{\sqrt{1-|a_{k_r}|^2}}{1-a_{k_r}\overline{z_A}} \prod_{t=0}^{k_r-1} \frac{\overline{z_A}-\overline{a_t}}{1-a_t\overline{z_A}} \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A} \prod_{s=0}^{k_{r+1}-1} \frac{z_A-a_s}{1-\overline{a_s}z_A} \frac{dz_A}{z_A} = 0.$$

For $|A| = 2$, (note that $k_r \neq k_{r+1}$, $k_s \neq k_{s+1}$ and $k_r = k_{s+1}$) we obtain

$$\begin{aligned} &\mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})\overline{\mathcal{B}_{k_s}(e^{ix_{j_A}})}\mathcal{B}_{k_{s+1}}(e^{ix_{j_A}}) \right] \\ &= \mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})\overline{\mathcal{B}_{k_s}(e^{ix_{j_A}})}\mathcal{B}_{k_r}(e^{ix_{j_A}}) \right] \\ &= \mathbb{E}_X \left[|\mathcal{B}_{k_r}(e^{ix_{j_A}})|^2\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})\overline{\mathcal{B}_{k_s}(e^{ix_{j_A}})} \right] \end{aligned}$$

$$= \frac{1}{2\pi i} \int_{|z_A|=1} \frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2} \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A} \prod_{s=0}^{k_{r+1}-1} \frac{z_A-a_s}{1-\overline{a_s}z_A} \tag{3.42}$$

$$\times \frac{\sqrt{1-|a_{k_s}|^2}}{1-a_{k_s}\overline{z_A}} \prod_{u=0}^{k_s-1} \frac{\overline{z_A}-\overline{a_u}}{1-a_u\overline{z_A}} \frac{\sqrt{1-|a_{k_{s+1}}|^2}}{1-\overline{a_{k_{s+1}}}z_A} \prod_{n=0}^{k_{s+1}-1} \frac{z_A-a_n}{1-\overline{a_n}z_A} \frac{dz_A}{z_A}.$$

On the one hand, if $k_s = k_{r+1}$ from expression (3.42) we obtain

$$
\begin{aligned}
\mathbb{E}_X\left[|\mathcal{B}_{k_r}(e^{ix_{j_A}})|^2|\mathcal{B}_{k_s}(e^{ix_{j_A}})|^2\right] &= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}z_A}\frac{1}{1-a_{k_r}\overline{z_A}}\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}z_A}\frac{1}{1-a_{k_s}\overline{z_A}}\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}z_A}\frac{1}{z_A-a_{k_s}}dz_A\\
&= \frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}a_{k_s}}\frac{a_{k_s}}{a_{k_s}-a_{k_r}}+\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}a_{k_r}}\frac{a_{k_r}}{a_{k_r}-a_{k_s}}.
\end{aligned}
$$

To estimate the last value we will consider the estimate by Lemma 3.2.5 and introduce the quantities $\epsilon_2 = |a_r - a_s|$ and ϵ_1 keeping in mind that $|a| < 1$. Thereby, we get

$$
\begin{aligned}
&\frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}a_{k_s}}\frac{a_{k_s}}{a_{k_s}-a_{k_r}}+\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}a_{k_r}}\frac{a_{k_r}}{a_{k_r}-a_{k_s}}\\
&\leq \sqrt{\frac{2}{\epsilon_1}-1}\frac{|a_1|}{\epsilon_2}+\sqrt{\frac{2}{\epsilon_1}-1}\frac{|a_2|}{\epsilon_2}\leq 2\sqrt{\frac{2}{\epsilon_1}-1}\frac{1}{\epsilon_2}=\frac{2}{\epsilon_2}\sqrt{\frac{2}{\epsilon_1}-1}.
\end{aligned}
$$

On the other hand, if $k_s \neq k_{r+1}$ from expression (3.42) we obtain

$$
\begin{aligned}
&\mathbb{E}_X\left[\left|\mathcal{B}_{k_r}(e^{ix_{j_A}})\right|^2\overline{\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})}\mathcal{B}_{k_s}(e^{ix_{j_A}})\right]\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-a_{k_{r+1}}\overline{z_A}}\prod_{t=0}^{k_{r+1}-1}\frac{\overline{z_A}-\overline{a_t}}{1-a_t\overline{z_A}}\frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}z_A}\prod_{u=0}^{k_s-1}\frac{z_A-a_u}{1-\overline{a_u}z_A}\frac{dz_A}{z_A}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{z_A-a_{k_{r+1}}}\\
&\times\prod_{t=0}^{k_{r+1}-1}\frac{\overline{z_A}-\overline{a_t}}{1-a_t\overline{z_A}}\frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}z_A}\prod_{u=0}^{k_s-1}\frac{z_A-a_u}{1-\overline{a_u}z_A}dz_A.
\end{aligned}
\tag{3.43}
$$

From the last expression (3.43) by assuming $k_s > k_{r+1}$ we get

$$
\begin{aligned}
&\mathbb{E}_X\left[|\mathcal{B}_{k_r}(e^{ix_{j_A}})|^2|\mathcal{B}_{k_s}(e^{ix_{j_A}})|^2\right]\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{z_A-a_{k_{r+1}}}\frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}z_A}\prod_{u=k_{r+1}}^{k_s-1}\frac{z_A-a_u}{1-\overline{a_u}z_A}dz_A\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{z_A-a_{k_{r+1}}}\\
&\times\frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}z_A}\frac{z_A-a_{k_{r+1}}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{u=k_{r+1}+1}^{k_s-1}\frac{z_A-a_u}{1-\overline{a_u}z_A}dz_A\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}z_A}\prod_{u=k_{r+1}+1}^{k_s-1}\frac{z_A-a_u}{1-\overline{a_u}z_A}dz_A
\end{aligned}
$$

$$= a_{k_r} \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}a_{k_r}} \frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}a_{k_r}} \prod_{u=k_{r+1}+1}^{k_s-1} \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}}. \tag{3.44}$$

We can estimate the absolute value of the last result in (3.44) as

$$\begin{aligned}
&|a_{k_r}| \left| \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}a_{k_r}} \frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}a_{k_r}} \prod_{u=k_{r+1}+1}^{k_s-1} \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| \\
&\leq \left| \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}a_{k_r}} \frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}a_{k_r}} \prod_{u=k_{r+1}+1}^{k_s-1} \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| \\
&\leq \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}a_{k_r}} \frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}a_{k_r}} \left| \prod_{u=k_{r+1}+1}^{k_s-1} \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| \\
&\leq \frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}a_{k_r}} \frac{\sqrt{1-|a_{k_s}|^2}}{1-\overline{a_{k_s}}a_{k_r}} \prod_{u=k_{r+1}+1}^{k_s-1} \left| \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| \\
&\leq \sqrt{\frac{1-|a_{k_{r+1}}|^2}{|1-\overline{a_{k_{r+1}}}a_{k_r}|^2}} \sqrt{\frac{1-|a_{k_s}|^2}{|1-\overline{a_{k_s}}a_{k_r}|^2}} \prod_{u=k_{r+1}+1}^{k_s-1} \left| \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| \leq \frac{2}{\epsilon} - 1,
\end{aligned}$$

such that

$$|a_{k_r}| < 1 \text{ and } d(a_{k_r}, a_u) = \left| \frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}} \right| < 1,$$

where $d(a_{k_r}, a_u)$ is called *pseudo-hyperbolic distance* between a_u and a_{k_r}.
Note that if $k_s < k_{r+1}$, we obtain basically the same result. For $|A| = 3$, (note that $k_r \neq k_{r+1}$, $k_s \neq k_{s+1}$, $k_t \neq k_{t+1}$ and $k_r = k_{t+1}$) we get

$$\begin{aligned}
&\mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ixj_A})} \mathcal{B}_{k_{r+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_s}(e^{ixj_A})} \mathcal{B}_{k_{s+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_t}(e^{ixj_A})} \mathcal{B}_{k_{t+1}}(e^{ixj_A}) \right] \\
&= \mathbb{E}_X \left[\overline{\mathcal{B}_{k_r}(e^{ixj_A})} \mathcal{B}_{k_{r+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_s}(e^{ixj_A})} \mathcal{B}_{k_{s+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_t}(e^{ixj_A})} \mathcal{B}_{k_r}(e^{ixj_A}) \right] \\
&= \mathbb{E}_X \left[\left| \mathcal{B}_{k_r}(e^{ixj_A}) \right|^2 \mathcal{B}_{k_{r+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_s}(e^{ixj_A})} \mathcal{B}_{k_{s+1}}(e^{ixj_A}) \overline{\mathcal{B}_{k_t}(e^{ixj_A})} \right].
\end{aligned}$$

Let us again assume that $k_{r+1} = k_{s+1}$ and $k_s = k_t$, such that

$$\begin{aligned}
&\mathbb{E}_X \left[\left| \mathcal{B}_{k_r}(e^{ixj_A}) \right|^2 \left(\mathcal{B}_{k_{r+1}}(e^{ixj_A}) \right)^2 \left(\overline{\mathcal{B}_{k_s}(e^{ixj_A})} \right)^2 \right] \\
&= \frac{1}{2\pi i} \int_{|z_A|=1} \frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2} \left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A} \prod_{s=0}^{k_{r+1}-1} \frac{z_A - a_s}{1-\overline{a_s}z_A} \right)^2 \\
&\qquad \left(\frac{\sqrt{1-|a_{k_s}|^2}}{1-a_{k_s}\overline{z_A}} \prod_{u=0}^{k_s-1} \frac{\overline{z_A}-\overline{a_u}}{1-a_u\overline{z_A}} \right)^2 \frac{dz_A}{z_A}.
\end{aligned} \tag{3.45}$$

From the last expression, if $k_s = k_{r+1}$ we obtain

$$\begin{aligned}
\mathbb{E}_X\left[|\mathcal{B}_{k_r}(e^{ix_{j_A}})|^2|\mathcal{B}_{k_s}(e^{ix_{j_A}})|^4\right] &= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\left(\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\right)^2\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}z_A}\frac{1}{1-a_{k_r}\overline{z_A}}\left(\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}z_A}\frac{1}{1-a_{k_s}\overline{z_A}}\right)^2\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{\left(1-|a_{k_s}|^2\right)^2}{(1-\overline{a_{k_s}}z_A)^2}\frac{z_A}{z_A-a_{k_s}}dz_A\\
&= \frac{1-|a_{k_r}|^2}{1-\overline{a_{k_r}}a_{k_s}}\frac{a_{k_s}^2}{a_{k_s}-a_{k_r}}+\left(\frac{1-|a_{k_s}|^2}{1-\overline{a_{k_s}}a_{k_r}}\frac{a_{k_r}}{a_{k_r}-a_{k_s}}\right)^2.
\end{aligned}$$

In case that $k_s \neq k_{r+1}$ from expression (3.45) we have to consider the cases when $k_s < k_{r+1}$ and $k_s > k_{r+1}$. Let we start with considering the first one, that is, $k_s < k_{r+1}$.

$$\mathbb{E}_X\left[\left|\mathcal{B}_{k_r}(e^{ix_{j_A}})\right|^2\left(\mathcal{B}_{k_{r+1}}(e^{ix_{j_A}})\right)^2\left(\overline{\mathcal{B}_{k_s}(e^{ix_{j_A}})}\right)^2\right]$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{t=0}^{k_{r+1}-1}\frac{z_A-a_t}{1-\overline{a_t}z_A}\frac{\sqrt{1-|a_{k_s}|^2}}{1-a_{k_s}\overline{z_A}}\prod_{u=0}^{k_s-1}\frac{\overline{z_A}-\overline{a_u}}{1-a_u\overline{z_A}}\right)^2\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi i}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{|1-a_{k_r}\overline{z_A}|^2}\left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{t=k_s}^{k_{r+1}-1}\frac{z_A-a_t}{1-\overline{a_t}z_A}\frac{\sqrt{1-|a_{k_s}|^2}}{1-a_{k_s}\overline{z_A}}\right)^2\frac{dz_A}{z_A}\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{t=k_s}^{k_{r+1}-1}\frac{z_A-a_t}{1-\overline{a_t}z_A}\frac{\sqrt{1-|a_{k_s}|^2}}{z_A-a_{k_s}}\right)^2 z_A dz_A\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{(z_A-a_{k_s})^2}{(1-\overline{a_s}z_A)^2}\\
&\quad\left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{t=k_s+1}^{k_{r+1}-1}\frac{z_A-a_t}{1-\overline{a_t}z_A}\right)^2\frac{1-|a_{k_s}|^2}{(z_A-a_{k_s})^2}z_A dz_A\\
&= \frac{1}{2\pi}\int_{|z_A|=1}\frac{1-|a_{k_r}|^2}{1-\overline{a}_{k_r}z_A}\frac{z_A}{z_A-a_{k_r}}\frac{1-|a_{k_s}|^2}{(1-\overline{a_s}z_A)^2}\left(\frac{\sqrt{1-|a_{k_{r+1}}|^2}}{1-\overline{a_{k_{r+1}}}z_A}\prod_{t=k_s+1}^{k_{r+1}-1}\frac{z_A-a_t}{1-\overline{a_t}z_A}\right)^2 z_A dz_A\\
&= \frac{1-|a_{k_s}|^2}{(1-\overline{a_s}a_{k_r})^2}\frac{1-|a_{k_{r+1}}|^2}{\left(1-\overline{a_{k_{r+1}}}a_{k_r}\right)^2}\prod_{t=k_s+1}^{k_{r+1}-1}\left|\frac{a_{k_r}-a_t}{1-\overline{a_t}a_{k_r}}\right|^2 a_{k_r}^2\\
&= \frac{1-|a_{k_s}|^2}{(1-\overline{a_s}a_{k_r})^2}a_{k_r}\frac{1-|a_{k_{r+1}}|^2}{\left(1-\overline{a_{k_{r+1}}}a_{k_r}\right)^2}a_{k_r} \le \left(\frac{2}{\epsilon}-1\right)\left(\frac{2}{\epsilon}-1\right) = \left(\frac{2}{\epsilon}-1\right)^2,
\end{aligned}$$

keeping in mind that

$$|a_{k_r}| < 1 \text{ and } d(a_{k_r}, a_u) = \left|\frac{a_{k_r}-a_u}{1-\overline{a_u}a_{k_r}}\right| < 1.$$

For $|A| = n$, we shall obtain in same way

$$\mathbb{E}\left[\prod_{s=1}^{n}\overline{\mathcal{B}_{k_{r_s}}(e^{ix_{j_s}})}\mathcal{B}_{k_{r_s+1}}(e^{ix_{j_s}})\right] \leq \left(\frac{2}{\epsilon}-1\right)^{n-1}.$$

Before we will show this we are going to give two remarks. First of all we can easily check that that if the components are all different, i.e.

$$\mathbb{E}\left[\prod_{s=1}^{n}\overline{\mathcal{B}_{k_{r_s}}(e^{ix_{j_s}})}\mathcal{B}_{k_{r_s+1}}(e^{ix_{j_s}})\right]$$

with $k_{r_s} \neq k_{r_t}$ when $s \neq t$ we get zero as a result. Furthermore, it is also easy to see that the highest value in the estimates happen when the pair $\overline{\mathcal{B}_{k_r}(e^{ix_{j_A}})}\mathcal{B}_{k_s}(e^{ix_{j_A}})$ repeats itself n times with $r_s \neq r_s + 1$.
This means that, although keeping $r \neq t$, the main problem is to estimate the values of the integral for the expectation value

$$\mathbb{E}_X\left[\left(\overline{\mathcal{B}_{a_r}(x_{s_A})}\mathcal{B}_{a_t}(x_{s_A})\right)^n\right] = \frac{1}{2\pi}\int_{|z_A|=1}\left(\overline{\mathcal{B}_{a_r}(x_{s_A})}\mathcal{B}_{a_t}(x_{s_A})\right)^n \frac{dz}{iz_A} \tag{3.46}$$

which leads to

$$\mathbb{E}_X\left[\left(\mathcal{B}_{a_r}(x_{s_A})\overline{\mathcal{B}_{a_t}(x_{s_A})}\right)^n\right] = \frac{1}{2\pi}\int_{|z_A|=1}\left(\frac{\sqrt{1-|a_r|^2}}{1-\overline{a_r}z_A}\frac{\sqrt{1-|a_t|^2}}{1-a_t\overline{z_A}}\right)^n\frac{dz_A}{iz_A}\text{ , with } r \neq t. \tag{3.47}$$

Let we start our estimate as follows

$$\left|\frac{1}{2\pi}\int_{|z|=1}\left(\frac{\sqrt{1-|a_r|^2}}{1-\overline{a_r}z}\frac{\sqrt{1-|a_t|^2}}{1-a_t\overline{z}}\right)^n\frac{dz}{iz}\right| \leq \frac{1}{2\pi}\int_{|z|=1}\left|\frac{\sqrt{1-|a_r|^2}}{1-\overline{a_r}z}\frac{\sqrt{1-|a_t|^2}}{1-a_t\overline{z}}\right|^n\left|\frac{dz}{iz}\right|$$

$$= \frac{1}{|2\pi i|}\int_{|z|=1}\left|\frac{\sqrt{1-|a_r|^2}}{1-\overline{a_r}z}\frac{\sqrt{1-|a_t|^2}}{1-a_t\overline{z}}\right|^n|dz| = \frac{1}{|2\pi i|}\int_{|z|=1}\left|\sqrt{\frac{1-|a_r|^2}{|1-\overline{a_r}z|^2}}\sqrt{\frac{1-|a_t|^2}{|1-a_t\overline{z}|^2}}\right|^n|dz|.$$

$$\leq \frac{1}{|2\pi i|}\max_{a_r,a_t\in\mathbb{D}}\max_{z\in\mathbb{T}}\left|\sqrt{\frac{1-|a_r|^2}{|1-\overline{a_r}z|^2}}\sqrt{\frac{1-|a_t|^2}{|1-a_t\overline{z}|^2}}\right|^n\int_{|z|=1}|dz|$$

$$\leq \left(\max_{a_r\in\mathbb{D}}\max_{z\in\mathbb{T}}\sqrt{\frac{1-|a_r|^2}{|1-\overline{a_r}z|^2}}\max_{a_t\in\mathbb{D}}\max_{z\in\mathbb{T}}\sqrt{\frac{1-|a_t|^2}{|1-a_t\overline{z}|^2}}\right)^n.$$

$$\leq \left(\max_{a_n\in\mathbb{D}}\max_{z_A\in\mathbb{T}}\frac{1-|a_n|^2}{|1-\overline{a_n}z|^2}\right)^{n/2}\left(\max_{a_n\in\mathbb{D}}\max_{z_A\in\mathbb{T}}\frac{1-|a_m|^2}{|1-a_m\overline{z}|^2}\right)^{n/2}$$

$$= \left(\max_{a_n\in\mathbb{D}}\max_{z_A\in\mathbb{T}}\frac{1-|a_n|^2}{|1-\overline{a_n}z|^2}\right)^n \leq \left(\frac{2}{\epsilon}-1\right)^n.$$

Thereby, using Lemma 3.2.5 we get

$$\mathcal{C}(\mathcal{A},T) := \sum_{r_1,r_2,\ldots,r_{2n}\in T, r_j\neq r_{j+1}} \prod_{A\in\mathcal{A}} \left(\frac{2}{\epsilon}-1\right)^{|A|} = \prod_{A\in\mathcal{A}} \left(\frac{2}{\epsilon}-1\right)^{|A|} T^{2n-|A|+1}$$

$$= \mathcal{K} \times \#\Big\{(r_1,r_2,\ldots,r_{2n})\in T^{2n} : r_j \neq r_{j+1}$$
$$\wedge \sum_{r\in A} \int_{|z_A|=1} \prod_{r=1}^{2n} \overline{\mathcal{B}_{a_r}(z_A)} \mathcal{B}_{a_{r+1}}(z_A) \frac{dz_A}{iz_A} \neq 0, \ \forall A\in\mathcal{A}\Big\}, \tag{3.48}$$

where

$$\mathcal{K} := \prod_{A\in\mathcal{A}} \left(\frac{2}{\epsilon}-1\right)^{|A|}. \tag{3.49}$$

Let us remark that taking into account our proof everything depends on the result of the integral

$$\int_{|z_A|=1} \prod_{r=1}^{2n} \overline{\mathcal{B}_{a_r}(z_A)} \mathcal{B}_{a_{r+1}}(z_A) \frac{dz_A}{iz_A}$$

for $\mathcal{A}\in P(2n,t)$. Here, the indices $(t_1,t_2,\ldots,t_{2n})\in T^{2n}$ are subjected to the $|A|=t$ constraints $\int_{|z_A|=1}\prod_{j=1}^{2n}\overline{\mathcal{B}_{a_{r_j}}(z_A)}\mathcal{B}_{a_{r_{j+1}}}(z_A)\frac{dz_A}{iz_A}\neq 0$ for all $A\in\mathcal{A}$. In particular, when the indices t_j are all different we have that the result is zero. This also is the principal point if one want to extend the proof to more general cases like frames. □

3.2.4 Analysis of $\mathbb{P}(E_k)$

We have to treat the term $\mathbb{P}(E_k)$ in (3.34) and (3.35). In the usual manner, let $\beta_m = \beta^{n/K_m}$, $m=1,\ldots,n$ be positive numbers satisfying

$$\sum_{m=1}^{n} \beta_m = a_1$$

and $K_m\in\mathbb{N}$, $m=1,\ldots,n$, some natural numbers. For $k\in D$ as before we have

$$\mathbb{P}(E_k) = \mathbb{P}\Big(\sum_{m=1}^{n} \big|\big((D^{-1}HR_T)^m \mathrm{sgn}(c)\big)_k\big| \geq a_1\Big)$$

$$= \sum_{m=1}^{n} \mathbb{P}\big(\big|\big((D^{-1}HR_T)^m \mathrm{sgn}(c)\big)_k\big|^{2K_m} \beta_m^{-2K_m} \geq 1\big) \tag{3.50}$$

$$\leq \sum_{m=1}^{n} \mathbb{E}[\big|\big((D^{-1}HR_T)^m \mathrm{sgn}(c)\big)_k\big|^{2K_m}]\beta_m^{-2K_m}, \tag{3.51}$$

where we now have $\beta_m^{-2K_m} = \beta^{-2n}$ for all m. Note, that (3.50) and (3.51) are obtained from (even when the expectation is infinite) $\mathbb{E}(X) = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$.

Thus, we obtain

$$\mathbb{P}(E_k) \leq \beta^{-2n} \sum_{m=1}^{n} \mathbb{E}[\left|\left((D^{-1}HR_T)^m \mathrm{sgn}(c)\right)_k\right|^{2K_m}] \tag{3.52}$$

and the condition $a_1 = \sum_{m=1}^{n} \beta_m$ reads as

$$a_1 = a = \sum_{m=1}^{n} \beta^{n/K_m} < 1.$$

The following lemma is the version for the Takenaka-Malmquist system of our usual lemma for the expectation value appearing in (3.52). The following proof is similar to the one of Lemma 3.2.7.

Lemma 3.2.8. *For $k \in D$ and $c \in \ell_2(D)$ with $\operatorname{supp} c = T$ we have*

$$\mathbb{E}_X \left[\left|\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_k\right|^{2K}\right] \leq \sum_{t=1}^{\min\{Km,N\}} \frac{N!}{(N-t)!} \sum_{\mathcal{A}\in P(2Km,t)} \mathcal{C}_D \times \mathcal{B}(\mathcal{A},T).$$

Hereby, we identify partitions of $[2Km]$ in $P(2Km,t)$ with partitions of $[2K] \times [m]$ with fixes k_r and k_{r+1} such that $k_r \neq k_{r+1}$ and $\mathcal{C}_D = \left(\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2\right)^{-1}$ and

$$\mathcal{B}(\mathcal{A},T) = \sum_{\substack{k_1^{(1)},k_2^{(1)},\ldots,k_m^{(1)} \in T \\ \vdots \\ k_1^{(2K)},k_2^{(2K)},\ldots,k_m^{(2K)} \in T \\ k_j^{(p)} \neq k_{j+1}^{(p)}\ j\in[m]}} \prod_{A\in\mathcal{A}} \left(\frac{2}{\epsilon}-1\right)^t$$

for $|\pm 1 - a_n| = \epsilon$ or $|\pm i - a_n| = \epsilon$.

Proof. Let be $\sigma := \operatorname{sgn}(c)$ and

$$\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_k = \sum_{\ell_1,\ldots,\ell_m=1}^{N} \sum_{\substack{k_1,\ldots,k_m \in T \\ k_j \neq k_{j+1},\ j=1,\ldots,m}} \sigma(k_m) \frac{\prod_{r=0}^{m} \overline{\mathcal{B}_{k_r}(x_{l_r})}\mathcal{B}_{k_{r+1}}(x_{l_r})}{\prod_{s=1}^{2m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s}(e^{ix_{l_s}})\right|^2}$$

with $k_0 := k$. Wherefore

$$\left|\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_{k_0}\right|^2$$

$$= \sum_{\ell_1^{(1)},\ldots,\ell_m^{(1)}=1}^{N} \sum_{\ell_1^{(2)},\ldots,\ell_m^{(2)}=1}^{N} \sum_{\substack{k_1^{(1)},k_2^{(1)},\ldots,k_m^{(1)}\in T\\ k_1^{(2)},k_2^{(2)},\ldots,k_m^{(2)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)},\ j\in[m],\ p=1,2}} \overline{\sigma(k_m^{(1)})}\sigma(k_m^{(2)})$$

$$\times \frac{\prod_{r=1}^{m}\overline{\mathcal{B}_{k_r^{(1)}}\left(x_{l_r}^{(1)}\right)}\mathcal{B}_{k_{r+1}^{(1)}}\left(x_{l_r}^{(1)}\right)}{\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(1)}}(e^{ix_{l_s}^{(1)}})\right|^2} \frac{\prod_{r=1}^{m}\overline{\mathcal{B}_{k_r^{(2)}}\left(x_{l_r}^{(2)}\right)}\mathcal{B}_{k_{r+1}^{(2)}}\left(x_{l_r}^{(2)}\right)}{\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(2)}}(e^{ix_{l_s}^{(2)}})\right|^2}$$

$$= \sum_{\ell_1^{(1)},\ldots,\ell_m^{(1)}=1}^{N} \sum_{\ell_1^{(2)},\ldots,\ell_m^{(2)}=1}^{N} \sum_{\substack{k_1^{(1)},k_2^{(1)},\ldots,k_m^{(1)}\in T\\ k_1^{(2)},k_2^{(2)},\ldots,k_m^{(2)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}\ j\in[m]\ p=1,2}} \overline{\sigma(k_m^{(1)})}\sigma(k_m^{(2)})$$

$$\times \frac{\prod_{r=1}^{m}\frac{\sqrt{1-|a_{k_r^{(1)}}|^2}}{1-a_{k_r^{(1)}}e^{-ix_{l_r}^{(1)}}}\prod_{q_1^{(1)}=0}^{k_r^{(1)}-1}\frac{e^{-ix_{l_r}^{(1)}}-\overline{a_{q_1^{(1)}}}}{1-a_{q_1^{(1)}}e^{-ix_{l_r}^{(1)}}}\frac{\sqrt{1-|a_{k_{r+1}^{(1)}}|^2}}{1-\overline{a_{k_{r+1}^{(1)}}}e^{ix_{l_r}^{(1)}}}\prod_{q_2^{(1)}=0}^{k_{r+1}^{(1)}-1}\frac{e^{ix_{l_r}^{(1)}}-a_{q_2^{(1)}}}{1-\overline{a_{q_2^{(1)}}}e^{ix_{l_r}^{(1)}}}}{\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(1)}}(e^{ix_{l_s}^{(1)}})\right|^2}$$

$$\times \frac{\prod_{r=1}^{m}\frac{\sqrt{1-|a_{k_r^{(2)}}|^2}}{1-a_{k_r^{(2)}}e^{-ix_{l_r}^{(2)}}}\prod_{s_1^{(2)}=0}^{k_r^{(2)}-1}\frac{e^{-ix_{l_r}^{(2)}}-\overline{a_{s_1^{(2)}}}}{1-a_{s_1^{(2)}}e^{-ix_{l_r}^{(2)}}}\frac{\sqrt{1-|a_{k_{r+1}^{(2)}}|^2}}{1-\overline{a_{k_{r+1}^{(2)}}}e^{ix_{l_r}^{(2)}}}\prod_{s_2^{(2)}=0}^{k_{r+1}^{(2)}-1}\frac{e^{ix_{l_r}^{(2)}}-a_{s_2^{(2)}}}{1-\overline{a_{s_2^{(2)}}}e^{ix_{l_r}^{(2)}}}}{\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(2)}}(e^{ix_{l_s}^{(2)}})\right|^2}$$

where $k_0^{(1)}=k_0^{(2)}=k_0=k$. Taking a $2K$th power yields

$$\left|\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_{k_0}\right|^{2K}$$

$$= \sum_{\substack{\ell_1^{(1)},\ldots,\ell_m^{(1)}=1\\ \vdots\\ \ell_1^{(2K)},\ldots,\ell_m^{(2K)}=1}} \sum_{\substack{k_1^{(1)},k_2^{(1)},\ldots,k_m^{(1)}\in T\\ \vdots\\ k_1^{(2K)},k_2^{(2K)},\ldots,k_m^{(2K)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}}} \overline{\sigma(k_m^{(1)})}\sigma(k_m^{(2)})\cdots\overline{\sigma(k_m^{(2K-1)})}\sigma(k_m^{(2K)})$$

$$\times \frac{\prod_{p=1}^{2K}\prod_{r=1}^{m}\overline{\mathcal{B}_{k_r^{(p)}}\left(x_{l_r}^{(p)}\right)}\mathcal{B}_{k_{r+1}^{(p)}}\left(x_{l_r}^{(p)}\right)}{\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2}$$

$$= \sum_{\substack{\ell_1^{(1)},\dots,\ell_m^{(1)}=1\\ \vdots \\ \ell_1^{(2K)},\dots,\ell_m^{(2K)}=1}} \sum_{\substack{k_1^{(1)},k_2^{(1)},\dots,k_m^{(1)}\in T\\ \vdots \\ k_1^{(2K)},k_2^{(2K)},\dots,k_m^{(2K)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}}} \overline{\sigma(k_m^{(1)})}\sigma(k_m^{(2)})\cdots\overline{\sigma(k_m^{(2K-1)})}\sigma(k_m^{(2K)})$$

$$\times \frac{\prod_{p=1}^{2K}\prod_{r=1}^{m} \frac{\sqrt{1-|a_{k_r^{(p)}}|^2}}{1-a_{k_r^{(p)}}e^{-ix_{l_r}^{(p)}}} \prod_{q_1^{(p)}=0}^{k_r^{(p)}-1} \frac{e^{-ix_{l_r}^{(p)}}-\overline{a_{q_1^{(p)}}}}{1-a_{q_1^{(p)}}e^{-ix_{l_r}^{(p)}}} \frac{\sqrt{1-|a_{k_{r+1}^{(p)}}|^2}}{1-\overline{a_{k_{r+1}^{(p)}}}e^{ix_{l_r}^{(p)}}} \prod_{q_2^{(p)}=0}^{k_{r+1}^{(p)}-1} \frac{e^{ix_{l_r}^{(p)}}-a_{q_2^{(p)}}}{1-\overline{a_{q_2^{(p)}}}e^{ix_{l_r}^{(1)}}}}{\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2}$$

where $k_0^{(p)}=k$, $p=1,\dots,2K$. Further, recall that $|\sigma(k)|=1$ on T . Taking the expected value $\mathbb{E}_X$ yields

$$\mathbb{E}_X\left[\left|((HR_T)^m\operatorname{sgn}c)_{k_0}\right|^{2K}\right] \leq \sum_{\substack{\ell_1^{(1)},\dots,\ell_m^{(1)}=1\\ \vdots \\ \ell_1^{(2K)},\dots,\ell_m^{(2K)}=1}} \sum_{\substack{k_1^{(1)},k_2^{(1)},\dots,k_m^{(1)}\in T\\ \vdots \\ k_1^{(2K)},k_2^{(2K)},\dots,k_m^{(2K)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}}}$$

$$\times\mathbb{E}_X\left[\frac{\prod_{p=1}^{2K}\prod_{r=1}^{m} \frac{\sqrt{1-|a_{k_r^{(p)}}|^2}}{1-a_{k_r^{(p)}}e^{-ix_{l_r}^{(p)}}} \prod_{q_1^{(p)}=0}^{k_r^{(p)}-1} \frac{e^{-ix_{l_r}^{(p)}}-\overline{a_{q_1^{(p)}}}}{1-a_{q_1^{(p)}}e^{-ix_{l_r}^{(p)}}} \frac{\sqrt{1-|a_{k_{r+1}^{(p)}}|^2}}{1-\overline{a_{k_{r+1}^{(p)}}}e^{ix_{l_r}^{(p)}}} \prod_{q_2^{(p)}=0}^{k_{r+1}^{(p)}-1} \frac{e^{ix_{l_r}^{(p)}}-a_{q_2^{(p)}}}{1-\overline{a_{q_2^{(p)}}}e^{ix_{l_r}^{(1)}}}}{\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2}\right].$$

$$= \sum_{\substack{\ell_1^{(1)},\dots,\ell_m^{(1)}=1\\ \vdots \\ \ell_1^{(2K)},\dots,\ell_m^{(2K)}=1}} \sum_{\substack{k_1^{(1)},k_2^{(1)},\dots,k_m^{(1)}\in T\\ \vdots \\ k_1^{(2K)},k_2^{(2K)},\dots,k_m^{(2K)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}}} \times\left(\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2\right)^{-1}$$

$$\times\mathbb{E}_X\left[\prod_{p=1}^{2K}\prod_{r=1}^{m} \frac{\sqrt{1-|a_{k_r^{(p)}}|^2}}{1-a_{k_r^{(p)}}e^{-ix_{l_r}^{(p)}}} \prod_{q_1^{(p)}=0}^{k_r^{(p)}-1} \frac{e^{-ix_{l_r}^{(p)}}-\overline{a_{q_1^{(p)}}}}{1-a_{q_1^{(p)}}e^{-ix_{l_r}^{(p)}}} \frac{\sqrt{1-|a_{k_{r+1}^{(p)}}|^2}}{1-\overline{a_{k_{r+1}^{(p)}}}e^{ix_{l_r}^{(p)}}} \prod_{q_2^{(p)}=0}^{k_{r+1}^{(p)}-1} \frac{e^{ix_{l_r}^{(p)}}-a_{q_2^{(p)}}}{1-\overline{a_{q_2^{(p)}}}e^{ix_{l_r}^{(1)}}}\right]. \tag{3.53}$$

(with equality if all the entries of σ are equal on T). Let us consider the expected value appearing in the sum. As in the proof of Lemma 3.2.7 we have to take into account that some of the indices $\ell_r^{(p)}$ might coincide. This affords to introduce some additional notation. Let $\left(\ell_r^{(p)}\right)_{r=1,\dots,m}^{p=1,\dots,2K} \subset \{1,\dots,N\}^{2Km}$ be some vector of indices and let $\mathcal{A}=(A_1,\dots,A_t)\,, A_i \subset \{1,\dots,m\}\times\{1,\dots,2K\}$ be a corresponding partition such that (r,p) and (r',p') are contained in the same block if and only if $\ell_r^{(p)}=\ell_{r'}^{(p')}$ may unambiguously write ℓ_A instead of $\ell_r^{(p)}$ if $(r,p)\in A$. Using that all ℓ_A for $A\in\mathcal{A}$ are different and that the x_{ℓ_A} are independent we may write the expectation in the sum in (3.53) as

$$\mathbb{E}_X\left[\prod_{p=1}^{2K}\prod_{r=1}^{m}\frac{\sqrt{1-|a_{k_r^{(p)}}|^2}}{1-a_{k_r^{(p)}}e^{-ix_{l_r}^{(p)}}}\prod_{q_1^{(p)}=0}^{k_r^{(p)}-1}\frac{e^{-ix_{l_r}^{(p)}}-\overline{a_{q_1^{(p)}}}}{1-a_{q_1^{(p)}}e^{-ix_{l_r}^{(p)}}}\frac{\sqrt{1-|a_{k_{r+1}^{(p)}}|^2}}{1-\overline{a_{k_{r+1}^{(p)}}}e^{ix_{l_r}^{(p)}}}\prod_{q_2^{(p)}=0}^{k_{r+1}^{(p)}-1}\frac{e^{ix_{l_r}^{(p)}}-a_{q_2^{(p)}}}{1-\overline{a_{q_2^{(p)}}}e^{ix_{l_r}^{(1)}}}\right]$$
$$=\prod_{A\in\mathcal{A}}\mathbb{E}_X\left[\prod_{(r,p)\in A}\frac{\sqrt{1-|a_{k_r^{(p)}}|^2}}{1-a_{k_r^{(p)}}e^{-ix_{l_A}^{(p)}}}\prod_{q_1^{(p)}=0}^{k_r^{(p)}-1}\frac{e^{-ix_{l_A}^{(p)}}-\overline{a_{q_1^{(p)}}}}{1-a_{q_1^{(p)}}e^{-ix_{l_A}^{(p)}}}\frac{\sqrt{1-|a_{k_{r+1}^{(p)}}|^2}}{1-\overline{a_{k_{r+1}^{(p)}}}e^{ix_{l_A}^{(p)}}}\prod_{q_2^{(p)}=0}^{k_{r+1}^{(p)}-1}\frac{e^{ix_{l_A}^{(p)}}-a_{q_2^{(p)}}}{1-\overline{a_{q_2^{(p)}}}e^{ix_{l_A}^{(1)}}}\right]$$

Note that if $A\in\mathcal{A}$ contains only one element then the last expression vanishes due to the condition $k_r^{(p)}\neq k_{r+1}^{(p)}$ Thus, we only need to consider partitions $\mathcal{A}$ in $P(2Km,t)$. Now we are able to rewrite the inequality in (3.53) as

$$\mathbb{E}_X\left[\left|\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_k\right|^{2K}\right]\leq\sum_{t=1}^{\min\{Km,N\}}\frac{N!}{(N-t)!}\sum_{\mathcal{A}\in P(2Km,t)}\mathcal{C}_D\,\mathcal{B}(\mathcal{A},T),$$

with $\mathcal{C}_D=\left(\prod_{q=1}^{2K}\prod_{s=1}^{m}\sum_{l_s=1}^{N}\left|\mathcal{B}_{k_s^{(q)}}(e^{ix_{l_s}^{(q)}})\right|^2\right)^{-1}$ and

$$\mathcal{B}(\mathcal{A},T)=\sum_{\substack{k_1^{(1)},k_2^{(1)},\dots,k_m^{(1)}\in T\\ \vdots\\ k_1^{(2K)},k_2^{(2K)},\dots,k_m^{(2K)}\in T\\ k_j^{(p)}\neq k_{j+1}^{(p)}\ j\in[m]}}\prod_{A\in\mathcal{A}}\left(\frac{2}{\epsilon}-1\right)^t \tag{3.54}$$

for $|(\pm1,0)-a_n|=\epsilon$ or $|(0,\pm1)-a_n|=\epsilon$ with a small ϵ.
This proves the lemma. □

From now the objective is complete the proof by joining all the established facts to finish the proof of Theorem 3.2.1. Consider the absolute value of the quantity $C(\mathcal{A},T)$ defined in (3.37) for A $\in P(2n,t)$. Here the indices $(k_1,\dots,k_{2n})\in T^{2n}$ are subjected to the $|A|=t$ linear constraints $\sum_{r\in A}\overline{\mathcal{B}_{k_r}}\mathcal{B}_{k_{r+1}}=0$ for all $A\in\mathcal{A}$. These constraints are independent except for

$\sum_{r=1}^{2n} \overline{\mathcal{B}_{k_r}} \mathcal{B}_{k_{r+1}} = 0$. Thus, from (3.48) and (3.49) we can estimate

$$|C(\mathcal{A}, T)| \leq \mathcal{K}^t \, |T|^{2n-t+1} \leq \mathcal{K}^t M^{2n-t+1}. \tag{3.55}$$

By Lemma 3.2.7 we obtain (note that in Theorem 3.2.1 T is not random, so $\mathbb{E} = \mathbb{E}_X$)

$$\begin{aligned}\mathbb{E}\left[\left\|D^{-1}H_0^n\right\|_F^2\right] &\leq \mathcal{C}_D \sum_{t=1}^{\min\{n,N\}} \frac{N!}{(N-t)!} \sum_{\mathcal{A}\in P(2n,t)} \mathcal{K}^t \, |T|^{2n-t+1} \\ &\leq \mathcal{C}_D M^{2n+1} \sum_{t=1}^{n} \left(\frac{N\mathcal{K}}{M}\right)^t S_2(2n,t),\end{aligned}$$

where $S_2(n,t) = |P(2n,t)|$ are the associated Stirling numbers of the second kind. Set $\theta = \frac{N\mathcal{K}}{M}$.

The definition of the functions F_n and G_n, (see (A.7) and (A.8) in Appendix A) and Markov's inequality yields

$$\mathbb{P}\left(\left\|\left(D^{-1}H_0\right)^n\right\|_F \geq \kappa\right) = \mathbb{P}\left(\left\|\left(D^{-1}H_0\right)^n\right\|_F^2 \geq \kappa^2\right) \leq \kappa^{-2}\,\mathbb{E}\left[\left\|\left(D^{-1}H_0\right)^n\right\|_F^2\right]$$

$$\leq \kappa^{-2}\,\mathcal{C}_D \mathcal{K}^{2n}\, M\, N^{2n} \theta^{-2n} F_{2n}(\theta) = \kappa^{-2}\,\mathcal{C}_D \mathcal{K}^{2n}\, M\, N^{2n} G_{2n}(\theta).$$

Let us note that from (3.24) we have $\kappa < 1$. So, in this way we have $\left\|D\left(D^{-1}H_0\right)^n D^{-1}\right\|_F \leq \kappa$ which implies $\left(I_T - \left(D^{-1}H_0\right)^n\right)$ is invertible by the von Neumann series. In the same way, $\mathcal{F}_{TX}^* \mathcal{F}_{TX} = D_T\left(I_T - D^{-1}H_0\right)$ is invertible. Hence, also $\mathcal{F}_{TX}$ is injective.

Let us now consider $P(E_k)$. By Lemma 3.2.8 we need to bound $B(\mathcal{A}, T)$ defined in (3.54), i.e., the number of vectors $(k_j^{(p)}) \in T^{2Km}$ satisfying $k_j^{(p)} = k_{j+1}^{(p)}$ for all $A \in \mathcal{A}$ with $\mathcal{A} \in P(2Km, t)$. These are t independent linear constraints. Therefore, the number of these indices is bounded from above by $|T|^{2Km-t} \leq M^{2Km-t}$. Thus, in same way, by taking $\theta = \frac{N\mathcal{K}}{M}$, we obtain

$$\mathbb{E}_X\left[\left|\left(\left(D^{-1}HR_T\right)^m \operatorname{sgn} c\right)_k\right|^{2K}\right] \leq \mathcal{C}_D \sum_{t=1}^{Km} (N\mathcal{K})^t\, S_2(2Km,t) M^{2Km-t} = \mathcal{C}_D\, M^{2Km} F_{2Km}(\theta).$$

Let us consider again $\mathbb{P}(\text{failure})$ as the probability that the exact reconstruction of f by ℓ_1-minimization fails.

By Lemma 3.2.4 and (3.35) we again obtain

$$\begin{aligned}\mathbb{P}(\text{failure}) &\leq \mathbb{P}\left(\{\mathcal{F}_{TX}\{\textit{is not injective}\} \cup \{\sup_{k\in T^c}|P_k| \geq 1\}\right) \\ &\leq \mathcal{C}_D \sum_{k\in D} \mathbb{P}(E_k) + \mathbb{P}\left(\left\|\left(D^{-1}H_0\right)^n\right\| \geq k\right) \\ &\leq \mathcal{C}_D\,|D|\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2}\,\mathcal{C}_D \mathcal{K}^{2n}\, M\, N^{2n} G_{2n}(\theta) \\ &= \mathcal{C}_D\left[|D|\beta^{-2n} \sum_{m=1}^{n} G_{2mK_m}(\theta) + \kappa^{-2} \mathcal{K}^{2n}\, M\, N^{2n} G_{2n}(\theta)\right]\end{aligned}$$

under the conditions

$$a_1 = a = \sum_{m=1}^{n} \beta^{n/K_m} < 1, \quad a_2 + a_1 = 1, \text{ i.e., } a_2 = 1 - a,$$

$$\frac{\kappa}{1-\kappa} \leq \frac{a_2}{1+a_1} M^{-3/2} = \frac{1-a}{1+a} M^{-3/2}.$$

The other proofs are now basically the same as in the previous chapters.

3.3 Applications

The main goal of this section is to present some numerical experiments using Takenaka-Malmquist systems. Since their main field of application is in the study of transfer functions in systems identification we choose some examples of such functions. To make comparison we use the following two examples of transfer functions from a recent thesis [75], $E(z) = e^{e^z}$ and $F(z) = \frac{0.247z^4 + 0.0355z^3}{0.3329z^2 - 1.2727z + 1}$.

But before this we take the Poisson kernel $P(\zeta, z_0) = \dfrac{r^2 - |z_0|^2}{2\pi r |z_0 - \zeta|^2}$ as an example. This function is of particular interest to us due to the fact that whenever $|z_0| \to 1$ we get a singularity at the point z_0 whose influence can be studied for different parameters z_0.

For the numerical calculations we use the Matlab toolbox $\ell 1$-Magic [18] which adopts a Linear Programming to minimize the ℓ_1-norm of our coefficients x subject to $y = Ax$ using the primal-dual interior point method (see, for instance [82]) with A being our sampling matrix.

Since $\ell 1$-Magic works with real-valued vectors we need to modify our (complex-valued) system. We rewrite our complex multiplication $(\alpha + \mathbf{i}\beta)(v + \mathbf{i}w) = a + \mathbf{i}b$ as a matrix-vector multiplication, i.e.

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

This allows us to rewrite the complex linear system in the form $\tilde{y} = \mathcal{M}\tilde{x}$ with

$$M = \begin{pmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{pmatrix}$$

and

$$\tilde{x} = \begin{pmatrix} \mathrm{Re}(x) \\ \mathrm{Im}(x) \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} \mathrm{Re}(y) \\ \mathrm{Im}(y) \end{pmatrix}.$$

We need to choose our points a_i for the Blaschke products. To this end we take a grid given by the points

$$\{z_{k\ell} = r_k e^{\mathbf{i}\frac{2\pi\ell}{2^{2k}}},\ \ell = 0, 1, \ldots, 2^{2k} - 1, k = 0, \ldots, m\}$$

where $r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}}$ denotes the radius of the concentric circles such that on the circle with radius r_k we take 2^{2k} equidistant points (see for instance [62]).

From this grid we take N randomly chosen points, i.e. a vector $\mathbf{a} = (a_1, \ldots, a_N)$.

For our examples we made the simulation using Matlab 8.5.0(R2015a) running on a laptop with Intel(R) Core(TM) i3-4010U CPU 1.70 GHz, RAM 4GB, Windows 10, OS 64-bit(win64).

Example 1. *Consider the Poisson kernel over the unit circle (see [49])*

$$g(\zeta, z_0) = \frac{1 - |z_0|^2}{2\pi |z_0 - \zeta|^2}.$$

In the first case we choose z_0 to be near to zero, in this case $z_0 = -0.1$ and the function is sampled by $N = 1300$ measurements. In Figure 3.1 we can see the reconstruction of the function by using only $M = 55$ samples. This corresponds to 4.23% of our total measurements.

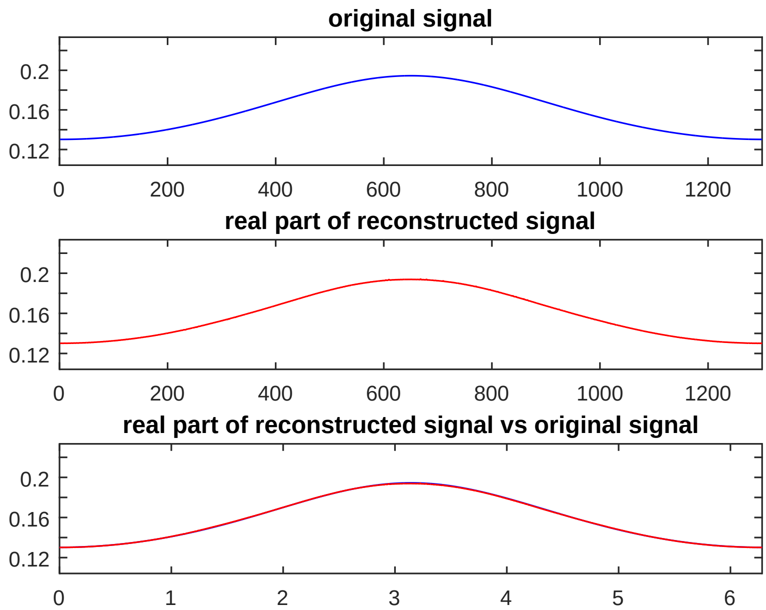

Figure 3.1: 55 sampling points corresponds to $\approx 4.23\%$ of total measurements. Time = 329.85 s, relative error = 0.0045.

In the case of $z_0 = -0.5$ with the same number of measurements ($N = 1300$) we can see a similar level of reconstruction (c.f. Figure 3.2) with a slight increase in the number of used samples ($M = 70$).

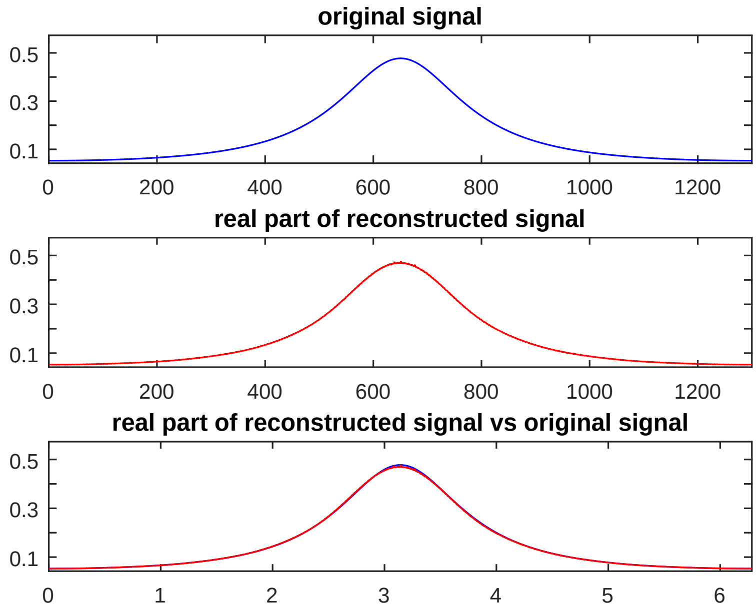

Figure 3.2: 70 sampling points corresponds to $\approx 5.38\%$ of total measurements. Time = 302.72 s, relative error = 0.0057.

In the case where the value of z_0 is near to the unit circle, i.e., $|z_0| \cong 1$ (in our case we choose $z_0 = -0.8$), with the same number of measurements ($N = 1300$) we can see that again with a slight increase in the number of taken sampling points ($M = 130$) we get a similar quality of reconstruction.

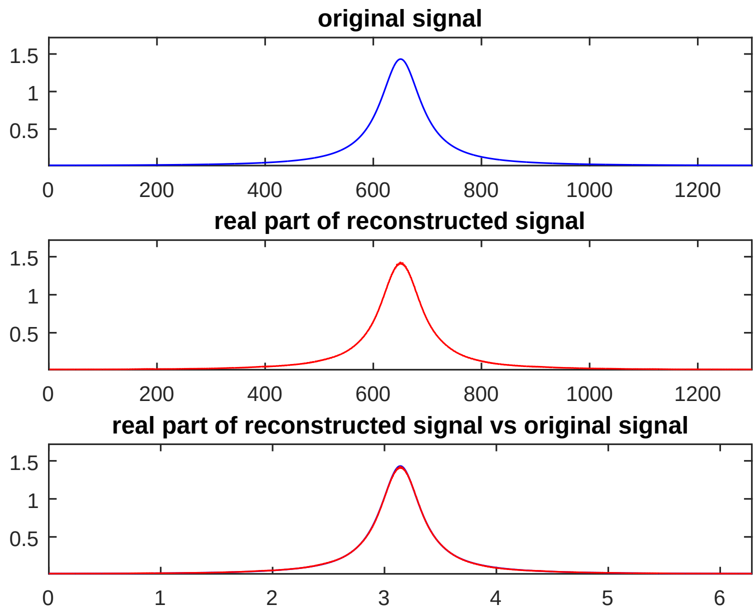

Figure 3.3: 130 sampling points equivalent to 10% of total measurements. Time = 324.75 s, relative error = 0.0094.

Using the same percentage (10%) but for a smaller number of sampling points ($M = 200$) in the case of $z_0 = -0.8$ we still can get a decent approximation with a dramatically smaller running time.

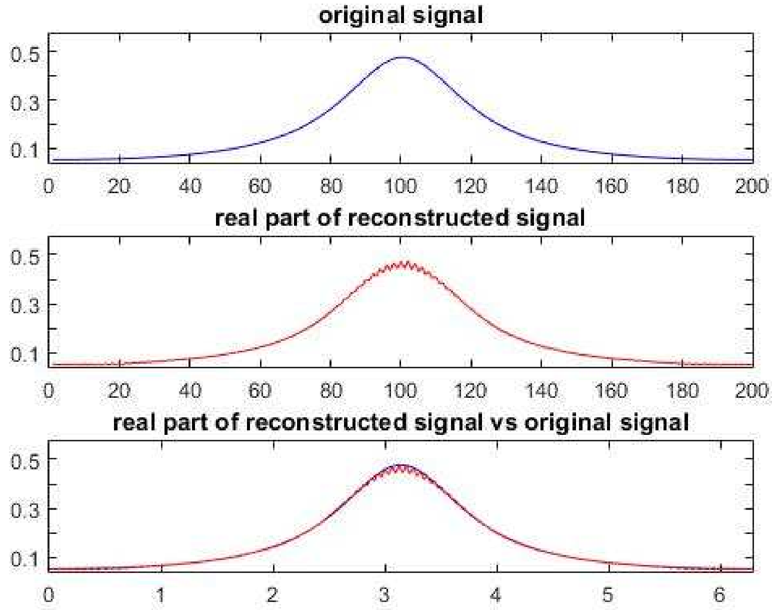

Figure 3.4: 20 sampling points corresponds to 10% of total measurements. Time = 2.0242 s, relative error = 0.0145.

From the Example 1 we make the following observations:

1. *Within the same number of measurements, when $|z_0|$ is near to zero we have the best reconstruction in the least time.*

2. *When the modulus of the parameter z_0 is close to 1 it requires more samples to reconstruct the signal.*

3. *The reconstruction is better in case when $|a_j - r| < \epsilon$ with ϵ relatively small and the parameter a_j being randomly chosen.*

We will use the next examples to compare our results with the results from the PhD-thesis of L. Shuang [75]. Note that in his case he chooses the parameter a_j from an a-priori given grid while in our case we use randomly chosen parameters.

Example 2. *Consider the example of the transfer function (from [75])*

$$F(z) = \frac{0.247z^4 + 0.0355z^3}{0.3329z^2 - 1.2727z + 1}.$$

For this example we sample the above function using 1000 samples (same as in [75]) .

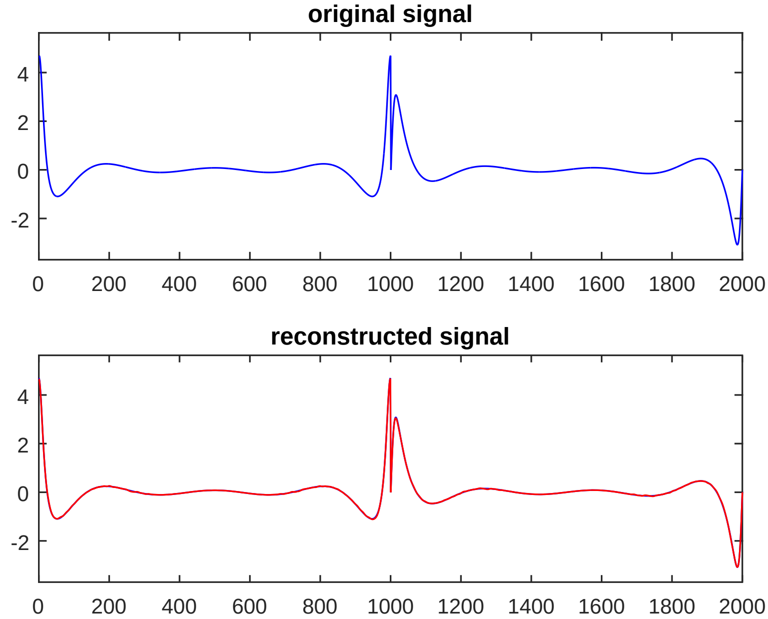

Figure 3.5: The original image and the reconstructed image obtained from 55 random samples $(5, 5\%)$, relative error = 3.2271e-04.

In Shuang's work the relative error is 0.022 *compared to the relative error of approximately* 0.0004 *in our case (c.f. Figure 3.5). Additionally, let us take a look to what happens if we take a bigger number of samples of the original function (*$M = 110$*).*

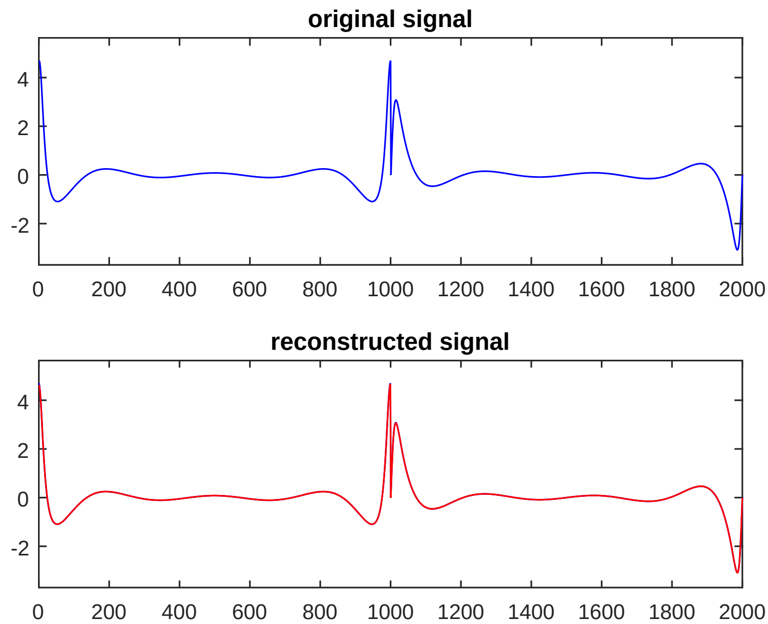

Figure 3.6: The original signal and the reconstructed signal obtained from 110 samples (11%), relative error = 1.8086e-04.

The original signal and the reconstructed signal can be seen in Figure 3.6. We can point out that the relative error is less than 0.0002 *compared to a relative error of* 0.004 *in Shuang's work.*

Unfortunately, since the author in [75] did not provide any information on the used hardware any comparison of runtimes is pure speculation.

Example 3. *Consider the function*

$$E(z) = e^{e^z}.$$

As we did before in Example 2 we sample our function in the same way (1000 samples) as in [75].

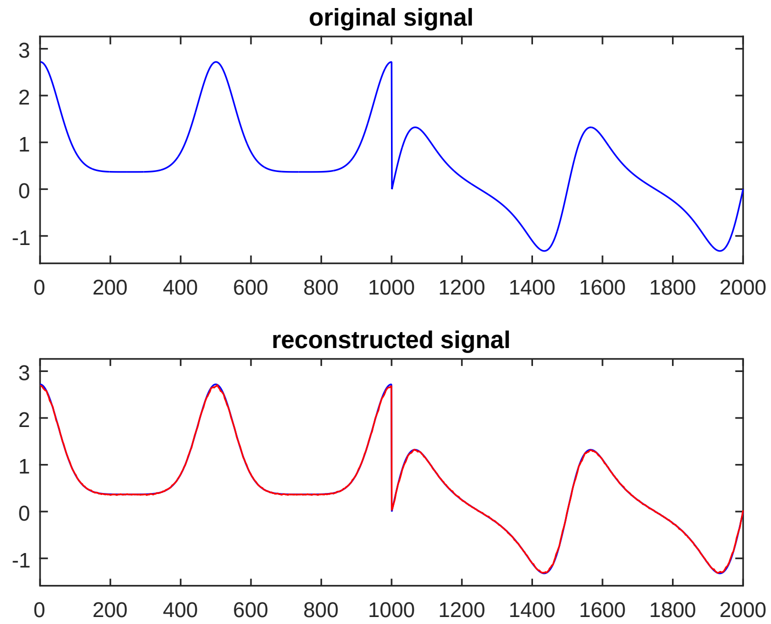

Figure 3.7: Reconstruction using 12 sampling points (corresponding to 1.2% of total measurements). Relative error = 4.5945e-04.

In Figure 3.7 we can observe that with only $M = 12$ *sampling points we can reconstruct our function with relative error* $= 0.00046$*. In the work of Shuang the reconstruction was done with a relative error of* 0.0004 *but using a larger number of sampling points (*$M = 55$*).*

In Figure 3.8 we can observe that if we use the same number of sampling points $M = 55$ *as in [75] then we get a relative error of* 0.00002*.*

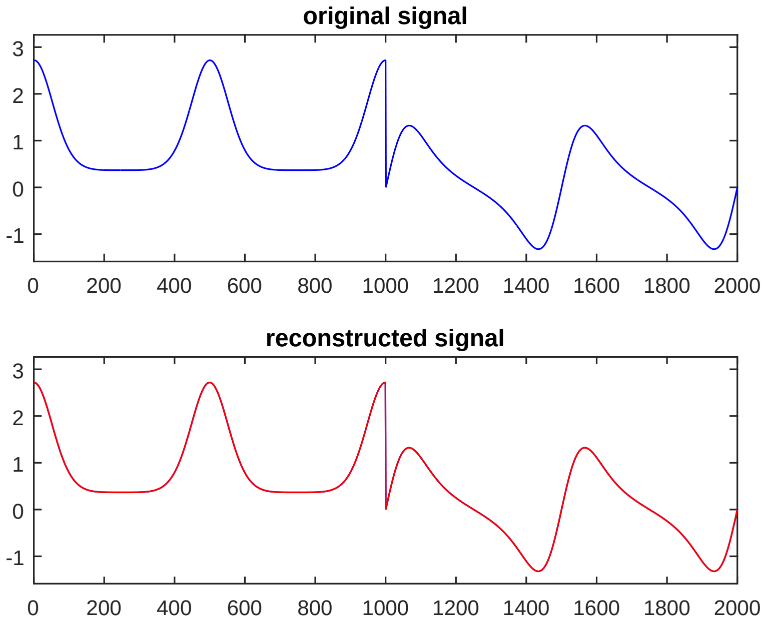

Figure 3.8: 55 sampling points corresponds to $5,5\%$ of total measurements. Relative error = 1.8646e-05

For $M = 110$ *(Figure 3.9) the relative error is* 0.000004 *in contrast to the relative error from Shuang example which is* 0.00003*.*

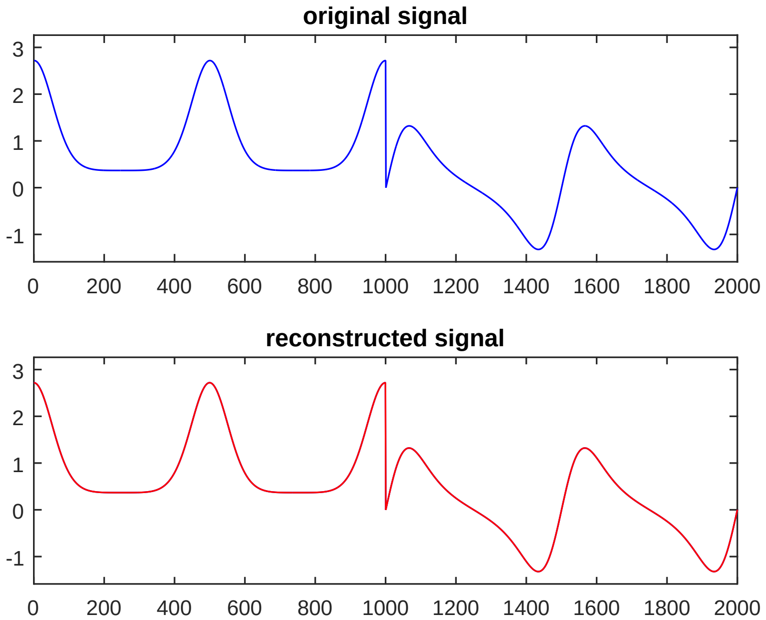

Figure 3.9: 110 sampling points corresponds to 11% of total measurements. Relative error = 3.8346e-06.

Again, since the author in [75] did not provide any information on the used hardware any *comparison of runtimes is pure speculation.*

Taking into account Example 2 and Example 3 we can make the following observations:

1. Using the same number of measurements our method provides a better approximation than the approach in the thesis of Shuang [75];

2. Moreover, the same relative error is attained with our method by using a a smaller number of sampling points.

Chapter 4

Conclusion and future research

> “ *- Would you tell me, please, which way I ought to go from here?*
> *- That depends a good deal on where you want to get to.* ”
>
> Lewis Carroll, *Alice’s Adventures in Wonderland*, 1865

In this thesis we study the question under which conditions compressed sensing techniques can be applied to sampling matrices coming from bicomplex and quaternionic sampling problems. Furthermore, we take a closer look into the possibility of using such techniques in the case of Takenaka-Malmquist systems. In contrast to the usual approach of checking the RIP condition and the incoherence of the dictionary, we base our investigation on a lemma by Candes, Romberg, and Tao. We show that the sampling matrices which appear in this context do allow the application of of compressed sensing with a certain probability. In all of these cases the principal problem lies in the calculation of the expectation value of the Frobenius norm of the powers of our sampling matrix. By using the idempotent decomposition in the bicomplex case we can almost directly use the classic approach by Rauhut. In the quaternionic case this idea runs into difficulties due to the non-commutativity of quaternionic multiplication. We overcome this by working directly with the coordinates which creates some long calculations, unfortunately. Furthermore, we also look at the Takenaka-Malmquist system. Here we clearly see the limitations of our approach which among other things uses the product property of the exponential and is therefore ideally suited for classic Fourier bases, but has problems in terms of general orthogonal bases without this product property. We overcome this by using more general estimates. Although the results are not as nice as in the case of bases based on classic Fourier bases we still get quantitative results. Here, we see the big advantage of this method. In difference to other works on compressed sensing techniques like the thesis of L. Shuang which can only get qualitative results (being based on asymptotic analysis) we do have quantitative estimates. Furthermore, we show the practical applicability of our approach in the case of quaternionic signals and Takenaka-Malmquist system. In the case of the quaternionic signal we study the reconstruction of color images where the color is represented by a quaternion. In the case of Takenaka-Malmquist systems we look at the sampling of transfer functions appearing in systems identification and compare it with the work of L. Shuang.

There are several points which require future research. First of all, in the case of the quaternionic signal the examples are being calculated by explicitly determining the sampling matrix which leads to rather inefficient algorithms, especially in terms of memory requirements. Here, the establishment of more efficient algorithms is required.

Furthermore, there are several cases which require additional research. Obviously, the most natural candidate would be the generalization to the case of Clifford-algebra valued functions. In contrast to the case of quaternions Clifford multiplication does not preserve the norm, i.e. we have $\|ab\| \leq 2^{n/2}\|a\|\|b\|$. A direct application leads to the inclusion of the constant in our estimates which cannot be simply dismissed. Here, more ideas are necessary as to modifying the estimates so that they still are meaningful. Another case to be studied is the case of slice-monogenic functions (see [26]).

Appendix A

Set Partition Theory

In this Appendices we will present some basics theory and necessary notations about set partitions (c.f. [11], [69]).

Definition A.0.1. *Set* $[n] := \{1, 2, \ldots, n\}$*. A partition of* $[n]$ *is a set of subsets of* $[n]$ *called blocks such that each* $r \in [n]$ *is contained in precisely one of the subsets. Let* $P(n, k)$ *denote the set of all partitions of* $[n]$ *into exactly* k *blocks such that each block contains at least* 2 *elements.*

Note that when $n < 2k$, $P(n, k)$ is empty.

Example 4. *Let* $n = 4$ *and* $k = 2$*. Then* $\{\{1, 2\}, \{3, 4\}\}$, $\{\{2, 3\}, \{1, 4\}\}$ *and* $\{\{1, 3\}, \{2, 4\}\}$ *corresponds the set of all partition of* $[4]$ *into exactly* 2 *blocks.*

Definition A.0.2. *The number of the all set partition* $S_2(n, k) = |P(n, k)|$ *are called associated Stirling numbers of the second kind.*

The exponential generating function ([70]) from Definition A.0.2 can be written as

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\lfloor n/2 \rfloor} S_2(n, k) y^k \frac{x^n}{n!} = e^{(y(e^x - x - 1))}. \tag{A.1}$$

From (A.1), the Stirling numbers of the second kind obey the recurrence relation

$$S_2(n, k) = kS_2(n - 1, k) + (n - 1)S_2(n - 2, k - 1). \tag{A.2}$$

For this work, in particular, we need a different kind of partitions. We define here that two numbers are called adjacent if they are consecutive in same block of a partition. Note that the last and the first numbers are also consecutive (in circular sense).
We denote the set $U(n, k)$ as the set of all partitions into k subsets having *no* adjacencies. Note that $k = 1$ is empty.

Example 5. *Let* $U(3, 2)$*. Then unique possible partition is* $\{\{1, 3\}, \{2\}\}$.

Example 6. *Let* $U(5, 3)$*. Then partitions are*

$$\{\{1, 3\}, \{2, 5\}, \{4\}\}, \quad \{\{1, 3\}, \{2, 4\}, \{5\}\}, \quad \{\{1, 4\}, \{2, 5\}, \{3\}\},$$

$$\{\{1,4\},\{3,5\},\{2\}\},\quad \{\{2,4\},\{3,5\},\{1\}\}.$$

We will present also a slight variation of the $U(n,k)$ partitions. Let $[K]\times[m]=\{1,\dots,K\}\times\{1,\dots,m\}$ for some numbers $K,m\in\mathbb{N}$. We denote by $U^*(K,m,s)$ the set of all partitions of $[K]\times[m]$ such that (p,u) and $(p,u+1)$ are not contained in the same block for all $p=1,\dots,K$ and $u=1,\dots,m-1$ (not considered as in circular sense). We notice here that $U(K,1,k)$ is the set of all partitions of a K-element set into k subsets (without any restriction on the type of partition). In particular, the numbers $|U(K,1,k)|$ equal the (ordinary) Stirling numbers $S(K,k)$ of the second kind. The numbers $b_n:=\sum_{k=1}^n S(n,k)$ are called Bell numbers.

We consider now $\mathcal{A}=\{A_1,\dots,A_t\}\in P(n,t)$ and $\mathcal{B}=\{A_1,\dots,A_s\}\in U(n,s)$. We associate a $t\times s$ matrix $M=M(\mathcal{A},\mathcal{B})$ to the pair $\mathcal{A},\mathcal{B}$ by setting

$$M_{i,j}:=|A_i\cap B_j|-|(A_i+1)\cap B_j|,\ \ 1\le i\le t,\ \ 1\le j\le s, \tag{A.3}$$

where A_i+1 is understood as the set whose elements are ones of A_i incremented by 1 in the circular sense, i.e., $n+1\equiv 1$.

Then we define $Q(n,t,s,R)$ to be the number of pairs of partitions $(\mathcal{A},\mathcal{B})$ with $\mathcal{A}\in P(n,t)$ and $\mathcal{B}\in U(n,s)$ such that the rank of $M(\mathcal{A},\mathcal{B})$ equals R, i.e.,

$$Q(n,t,s,R):=\#\{(\mathcal{A},\mathcal{B})\ :\ \mathcal{A}\in P(n,t),\ \mathcal{B}\in U(n,s),\ \operatorname{rank} M(\mathcal{A},\mathcal{B})=R\}. \tag{A.4}$$

Note that if A_i's are disjoint then

$$\sum_{i=1}^t M_{i,j}=\sum_{i=1}^t(|A_i\cap B_j|-|(A_i+1)\cap B_j|)=|\{1,\dots,n\}\cap B_j|-\{1,\dots,n\}\cap B_j|=0.$$

And in the same way, $\sum_{i=1}^t M_{i,j}=0$. Thus, the rank of $M(\mathcal{A},\mathcal{B})$ is less or equal to $\min s,t-1$. In other words $Q(n,t,s,R)=0$ if $R\ge\min s,t$.

Similarly, let $(\mathcal{A},\mathcal{B})$ be a pair of partitions of $[K]\times[m]$ where $\mathcal{A}=A_1,\dots,A_t\in P(Km,t)$ (identifying $[K]$ with $[K]\times[m]$) and $\mathcal{B}=B_1,\dots,B_s\in U^*(K,m,s)$. Let A_i-1 denote the sets whose elements are $\{(p,u-1),(p,u)\in A_i\}$. In contrast to above we do not calculate in the circular sense this time. So elements of the form $(p,0)$ may appear in A_i-1. Then to such a pair $(\mathcal{A},\mathcal{B})$ we associate a matrix $L=L(\mathcal{A},\mathcal{B})$ with entries

$$L_{i,j}=\sum_{(p,u)\in A_i\cap B_j}(-1)^p-\sum_{(p,u)\in(A_i-1)\cap B_j}(-1)^p. \tag{A.5}$$

Let us also define

$$Q^*(K,m,t,s,R):=\#\{(\mathcal{A},\mathcal{B})\ :\ \mathcal{A}\in P(n,t),\ \mathcal{B}\in U^*(K,m,s),\ \operatorname{rank} L(\mathcal{A},\mathcal{B})=R\}. \tag{A.6}$$

Let $F_n(\theta)$, $n \in \mathbb{N}$ denote the functions defined in terms of a generating function by

$$\sum_{n=1}^{\infty} F_n(\theta)\frac{x^n}{n!} = e^{\theta(e^x - x - 1)}. \tag{A.7}$$

In addition, we define

$$G_n(\theta) := \theta^{-n} F_n(\theta), \tag{A.8}$$

where is obvious that F_n is connected to the Stirling numbers of the second kind $S_2(n, k)$ from (A.1).

Appendix B

Calculation of $\left(|\left((HR_T)^m\sigma\right)_{k\ell}|^2\right)^K$

In this appendix we present the detailed calculation of $\left(|\left((HR_T)^m\sigma\right)_{k\ell}|^2\right)^K$ from the proof of Lemma 2.2.9.

$$((HR_T)\,[(k_1,\ell_1),(k_{m+1},\ell_{m+1})])^m =$$

$$= \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} HR_T\,[(k_1,\ell_1),(k_2,\ell_2)]\cdots HR_T\,[(k_m,\ell_m)\,(k_{m+1},\ell_{m+1})]$$

$$= \sum_{\substack{i_1=1 \\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \left[\prod_{r=1}^{m} e^{-\mathbf{J}\ell_r y_{i_r}} e^{\mathbf{I}(k_{r+1}-k_r)x_{i_r}} e^{\mathbf{J}\ell_{r+1}y_{i_r}}\right]$$

$$= \sum_{\substack{i_1=1 \\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\underline{a}\in\{-1,1\}^m} e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \prod_{s=1}^{m} \lambda(a_s,\theta_s)$$

$$= \sum_{\substack{i_1=1 \\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T \\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\lambda\left(\prod_{j=1}^{m} a_j, \theta_1 + \sum_{r=2}^{m}\alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\lambda\left(\prod_{j=1}^m a_j, \theta_1+\sum_{r=2}^m \alpha_r\theta_r\right).$$

By according the Lemma (2.2.6), we have

$$((HR_T)\left[(k_1,\ell_1),(k_{m+1},\ell_{m+1})\right])^m$$

$$= \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$e^{\mathbf{J}\sum_{r=1}^m \beta_r\phi_r} \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)\mathbf{I}$$

$$= \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r\right)+\mathbf{J}\sin\left(\sum_{r=1}^m \beta_r\phi_r\right)\right) \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^m \alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\left(\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)+\mathbf{J}\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_n^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{I}$$

$$= \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{J}$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{I}$$

$$+ \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\mathbf{JI}$$

$$= P_0+\mathbf{I}P_1+\mathbf{J}P_2-\mathbf{K}P_3.$$

Let us consider $\sigma_m(k_{m+1},\ell_{m+1})=\sigma_m(k_1,\ell_1)=\sigma_{0m}(k_1,\ell_1)+\mathbf{I}\sigma_{1m}(k_1,\ell_1)+\mathbf{J}\sigma_{2m}(k_1,\ell_1)+\mathbf{K}\sigma_{3m}(k_1,\ell_1)$. We are interested to compute

$$\sum_{(k_1,\ell_1)\in T}[P_0+\mathbf{I}P_1+\mathbf{J}P_2-\mathbf{K}P_3]\times[\sigma_{0m}(k_1,\ell_1)+\mathbf{I}\sigma_{1m}(k_1,\ell_1)+\mathbf{J}\sigma_{2m}(k_1,\ell_1)+\mathbf{K}\sigma_{3m}(k_1,\ell_1)]$$

$$
\begin{aligned}
= \sum_{(k_1,\ell_1)\in T} \Big[&(P_0\sigma_{0m}(k_1,\ell_1) - P_1\sigma_{1m}(k_1,\ell_1) - P_2\sigma_{2m}(k_1,\ell_1) + P_3\sigma_{3m}(k_1,\ell_1)) \\
&+ (P_0\sigma_{1m}(k_1,\ell_1) + P_1\sigma_{0m}(k_1,\ell_1) + P_2\sigma_{\rho_0 3} + P_3\sigma_{2m}(k_1,\ell_1))\,\mathbf{I} \\
&+ (P_0\sigma_{2m}(k_1,\ell_1) - P_1\sigma_{3m}(k_1,\ell_1) + P_2\sigma_{0m}(k_1,\ell_1) - P_3\sigma_{1m}(k_1,\ell_1))\,\mathbf{J} \\
&+ (P_0\sigma_{03m}(k_1,\ell_1) + P_1\sigma_{2m}(k_1,\ell_1) - P_2\sigma_{1m}(k_1,\ell_1) - P_3\sigma_{0m}(k_1,\ell_1))\,\mathbf{K} \Big].
\end{aligned}
$$

$$
\begin{aligned}
= &\sum_{(k_1,\ell_1)\in T} (P_0\sigma_{0m}(k_1,\ell_1) - P_1\sigma_{1m}(k_1,\ell_1) - P_2\sigma_{2m}(k_1,\ell_1) + P_3\sigma_{3m}(k_1,\ell_1)) \\
&+ \sum_{(k_1,\ell_1)\in T} (P_0\sigma_{1m}(k_1,\ell_1) + P_1\sigma_{0m}(k_1,\ell_1) + P_2\sigma_{\rho_0 3} + P_3\sigma_{2m}(k_1,\ell_1))\,\mathbf{I} \\
&+ \sum_{(k_1,\ell_1)\in T} (P_0\sigma_{2m}(k_1,\ell_1) - P_1\sigma_{3m}(k_1,\ell_1) + P_2\sigma_{0m}(k_1,\ell_1) - P_3\sigma_{1m}(k_1,\ell_1))\,\mathbf{J} \\
&+ \sum_{(k_1,\ell_1)\in T} (P_0\sigma_{03m}(k_1,\ell_1) + P_1\sigma_{2m}(k_1,\ell_1) - P_2\sigma_{1m}(k_1,\ell_1) - P_3\sigma_{0m}(k_1,\ell_1))\,\mathbf{K}
\end{aligned}
$$

$$
= Q_0 + Q_1\mathbf{I} + Q_2\mathbf{J} + Q_3\mathbf{K}.
$$

Then,

$$
\begin{aligned}
Q_0 = \sum_{(k_1,\ell_1)\in T} \Bigg[&\sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \; \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \; \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m} \\
&\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right) \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{0m}(k_1,\ell_1) \\
&- \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \; \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \; \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m} \\
&\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right) \sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}} \alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{1m}(k_1,\ell_1) \\
&- \sum_{\substack{i_1=1\\ \vdots \\ i_m=1}}^{N} \; \sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}} \; \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}
\end{aligned}
$$

$$
\begin{aligned}
&\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{2m}(k_1,\ell_1)\\
&+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\quad\sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}}\quad\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}}2^{1-m}\\
&\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{3m}(k_1,\ell_1)\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
Q_1=\sum_{(k_1,\ell_1)\in T}\Bigg[&\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\quad\sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}}\quad\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}}2^{1-m}\\
&\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{1m}(k_1,\ell_1)\\
&+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\quad\sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}}\quad\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}}2^{1-m}\\
&\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{0m}(k_1,\ell_1)\\
&+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\quad\sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}}\quad\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}}2^{1-m}\\
&\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\ldots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{3m}(k_1,\ell_1)\\
&+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\quad\sum_{\substack{(k_2,\ell_2),\ldots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\ldots,m}}\quad\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}}2^{1-m}
\end{aligned}
$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{2m}(k_1,\ell_1)\Bigg]$$

$$Q_2=\sum_{(k_1,\ell_1)\in T}\Bigg[\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{2m}(k_1,\ell_1)$$

$$-\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{3m}(k_1,\ell_1)$$

$$+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{0m}(k_1,\ell_1)$$

$$-\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{1m}(k_1,\ell_1)\Bigg].$$

$$Q_3=\sum_{(k_1,\ell_1)\in T}\Bigg[\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{3m}(k_1,\ell_1)$$

$$+\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\ \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{2m}(k_1,\ell_1)$$

$$-\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\ \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=+1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\cos\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{1m}(k_1,\ell_1)$$

$$-\sum_{\substack{i_1=1\\ \vdots\\ i_m=1}}^{N}\ \sum_{\substack{(k_2,\ell_2),\dots,(k_m,\ell_m)\in T\\ (k_j,\ell_j)\neq(k_{j+1},\ell_{j+1}),\ j=1,\dots,m}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1\cdots a_m=-1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r\phi_r\right)\sum_{\alpha_2,\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2}\cdots\alpha_m^{\delta_m}\sin\left(\theta_1+\sum_{r=2}^{m}\alpha_r\theta_r\right)\sigma_{0m}(k_1,\ell_1)\Bigg].$$

To compute the square modulus of previous quantity $\left((HR_T)^m\sigma\right)_{k\ell}$ follows $|\left((HR_T)^m\sigma\right)_{k\ell}|^2 = Q_0^2+Q_1^2+Q_2^2+Q_3^2$. Thus, follows

$$Q_0^2$$

$$=\Bigg[\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)$$

$$-\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)$$

$$-\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$+\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}2^{1-m}$$

$$\left.\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right]$$

$$\times\left[\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\ \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\ \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}\right.$$

$$\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\ \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\ \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)$$

$$-\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\ \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\ \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)$$

$$+\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\ \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\ \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\Bigg].$$

$$Q_1^2$$

$$\begin{aligned}
=\Bigg[&\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}2^{1-m}\\
&\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\\
+&\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}2^{1-m}\\
&\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\\
+&\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}2^{1-m}\\
&\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\\
+&\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\ \sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}2^{1-m}\\
&\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\Bigg]\\
\times\Bigg[&\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\ \sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N}\ \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\ \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}\\
&\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)
\end{aligned}$$

$$+\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$+\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg].$$

$$Q_2^2$$

$$=\Bigg[\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N}\sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$-\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N}\sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)$$

$$+\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N}\sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)$$

$$-\sum_{(k_1^{(1)},\ell_1^{(1)})\in T}\sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N}\sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}2^{1-m}$$

$$\left.\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r\right)\sum_{\alpha_2^{(1)},\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\right]$$

$$\times\left[\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}\right.$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)$$

$$-\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}}2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sum_{(k_1^{(2)},\ell_1^{(2)})\in T}\sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N}\sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}}2^{1-m}$$

$$\left.\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r\right)\sum_{\alpha_2^{(2)},\dots,\alpha_m\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)\right].$$

Q_3^2

$$
\begin{aligned}
= \Bigg[& \sum_{(k_1^{(1)},\ell_1^{(1)})\in T} \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m} \\
& \cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)} \cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell) \\
+ & \sum_{(k_1^{(1)},\ell_1^{(1)})\in T} \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m} \\
& \cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)} \sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell) \\
- & \sum_{(k_1^{(1)},\ell_1^{(1)})\in T} \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m} \\
& \sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)} \cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell) \\
- & \sum_{(k_1^{(1)},\ell_1^{(1)})\in T} \sum_{i_1^{(1)},\ldots,i_m^{(1)}=1}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m} \\
& \sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)} \sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\Bigg] \\
\times\Bigg[& \sum_{(k_1^{(2)},\ell_1^{(2)})\in T} \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N} \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m} \\
& \cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell) \\
+ & \sum_{(k_1^{(2)},\ell_1^{(2)})\in T} \sum_{i_1^{(2)},\ldots,i_m^{(2)}=1}^{N} \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m} \\
& \cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)
\end{aligned}
$$

$$- \sum_{(k_1^{(2)},\ell_1^{(2)})\in T} \sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N} \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}$$

$$\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)$$

$$- \sum_{(k_1^{(2)},\ell_1^{(2)})\in T} \sum_{i_1^{(2)},\dots,i_m^{(2)}=1}^{N} \sum_{\substack{(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}$$

$$\left.\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)\right].$$

We can still write our previous huge expression as

$$Q_0^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\ i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)$$

$$\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)$$

$$- \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)$$

$$\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)$$

$$- \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)$$

$$\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)$$

$$\times\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\Bigg).$$

$$Q_1^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}$$

$$
\begin{aligned}
&\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell) \\
&+ \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell) \\
&+ \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell) \\
&+ \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\Bigg) \\
&\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell) \\
&+ \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)
\end{aligned}
$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)$$

$$\times\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg).$$

$$Q_2^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)$$

$$times\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)$$

$$\times\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)$$

$$\times\sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)$$

$$
\begin{aligned}
& - \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = -1}} 2^{1-m} \sin\left(\sum_{r=1}^{m} \beta_r^{(1)} \phi_r\right) \\
& \times \sum_{\alpha_2^{(1)},\dots,\alpha_m \in\{-1,1\}} \alpha_2^{\delta_2(1)} \cdots \alpha_m^{\delta_m(1)} \sin\left(\theta_1^{(1)} + \sum_{r=2}^{m} \alpha_r^{(1)} \theta_r^{(1)}\right) \sigma_{1m}^{(1)}(k,\ell)\Bigg) \\
& \times \Bigg(\sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = +1}} 2^{1-m} \cos\left(\sum_{r=1}^{m} \beta_r^{(2)} \phi_r^{(2)}\right) \\
& \times \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)} \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \cos\left(\theta_1^{(2)} + \sum_{r=2}^{m} \alpha_r^{(2)} \theta_r^{(2)}\right) \sigma_{2m}^{(2)}(k,\ell) \\
& - \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = -1}} 2^{1-m} \cos\left(\sum_{r=1}^{m} \beta_r^{(2)} \phi_r^{(2)}\right) \\
& \times \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)} \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \sin\left(\theta_1^{(2)} + \sum_{r=2}^{m} \alpha_r^{(2)} \theta_r^{(2)}\right) \sigma_{3m}^{(2)}(k,\ell) \\
& + \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = +1}} 2^{1-m} \sin\left(\sum_{r=1}^{m} \beta_r^{(2)} \phi_r^{(2)}\right) \\
& \times \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)} \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \cos\left(\theta_1^{(2)} + \sum_{r=2}^{m} \alpha_r^{(2)} \theta_r^{(2)}\right) \sigma_{0m}^{(2)}(k,\ell) \\
& - \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = -1}} 2^{1-m} \sin\left(\sum_{r=1}^{m} \beta_r^{(2)} \phi_r\right) \\
& \times \sum_{\alpha_2^{(2)},\dots,\alpha_m \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \sin\left(\theta_1^{(2)} + \sum_{r=2}^{m} \alpha_r^{(2)} \theta_r^{(2)}\right) \sigma_{1m}^{(2)}(k,\ell)\Bigg).
\end{aligned}
$$

$$
Q_3^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}) \in T \\ (k_1^{(2)},\ell_1^{(2)}) \in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)} = 1 \\ i_1^{(2)},\dots,i_m^{(2)} = 1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)}) \in T \\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)}) \in T \\ (k_j^{(1)},\ell_j^{(1)}) \neq (k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)}) \neq (k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}
$$

$$
\begin{aligned}
&\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell) \\
&\quad + \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell) \\
&\quad - \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell) \\
&\quad - \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} 2^{1-m}\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right) \\
&\times \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\Bigg) \\
&\quad \times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell) \\
&\quad + \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} 2^{1-m}\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = +1}} 2^{1-m} \sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \cos\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell) \\
&- \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(2)} \cdots a_m^{(2)} = -1}} 2^{1-m} \sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right) \\
&\times \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in\{-1,1\}} \alpha_2^{\delta_2(2)} \cdots \alpha_m^{\delta_m(2)} \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)\Bigg).
\end{aligned}
$$

Now we will write each expression still more simplified. Then follows

$$
\begin{aligned}
Q_0^2 = &\sum_{\substack{(k_1^{(1)},\ell_1^{(1)}) \in T \\ (k_1^{(2)},\ell_1^{(2)}) \in T}} \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)} = 1 \\ i_1^{(2)},\ldots,i_m^{(2)} = 1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)}) \in T \\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)}) \in T \\ (k_j^{(1)},\ell_j^{(1)}) \neq (k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)}) \neq (k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m} \\
&\Bigg(\sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = +1}} \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in\{-1,1\}} \alpha_2^{\delta_2(1)} \cdots \alpha_m^{\delta_m(1)} \\
&\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\right. \\
&\left.- \sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\right) \\
&- \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = -1}} \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in\{-1,1\}} \alpha_2^{\delta_2(1)} \cdots \alpha_m^{\delta_m(1)} \\
&\left(\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\right. \\
&\left.- \sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right)\Bigg)
\end{aligned}
$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Bigg(\sum_{r=1}^m \beta_r\phi_r^{(2)}\Bigg)\cos\Bigg(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\Bigg)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sin\Bigg(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Bigg)\cos\Bigg(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\Bigg)\sigma_{2m}^{(2)}(k,\ell)\Bigg)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Bigg(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Bigg)\sin\Bigg(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Bigg)\sigma_{1m}^{(2)}(k,\ell)$$

$$-\sin\Bigg(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Bigg)\sin\Bigg(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\Bigg)\sigma_{3m}^{(2)}(k,\ell)\Bigg)\Bigg).$$

$$Q_1^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\Bigg(\sum_{r=1}^m \beta_r\phi_r^{(1)}\Bigg)\cos\Bigg(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\Bigg)\sigma_{1m}^{(1)}(k,\ell)$$

$$+\sin\Bigg(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Bigg)\cos\Bigg(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\Bigg)\sigma_{3m}^{(1)}(k,\ell)\Bigg)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} \sum_{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\Bigg(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Bigg)\sin\Bigg(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Bigg)\sigma_{0m}^{(1)}(k,\ell)$$

$$+\sin\Bigg(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Bigg)\sin\Bigg(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\Bigg)\sigma_{2m}^{(1)}(k,\ell)\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\Bigg)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg)\Bigg).$$

$$Q_2^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\ i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}} \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\Bigg)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}} \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\right.$$

$$\left.+\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)\right)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\right.$$

$$\left.\left.+\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)\right)\right).$$

$$Q_3^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T \\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1 \\ i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1}} \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right.$$

$$\left.-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\right)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1}} \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\left(\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\right.$$

$$\left.-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\right)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\ \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m\beta_r\phi_r^{(2)}\Big)\cos\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{3m}^{(2)}(k,\ell)$$

$$-\sin\Big(\sum_{r=1}^m\beta_r^{(2)}\phi_r^{(2)}\Big)\cos\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{1m}^{(2)}(k,\ell)\Bigg)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\ \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m\beta_r^{(2)}\phi_r^{(2)}\Big)\sin\Big(\sum_{r=1}^m\beta_r^{(2)}\phi_r^{(2)}\Big)\sigma_{2m}^{(2)}(k,\ell)$$

$$-\sin\Big(\sum_{r=1}^m\beta_r^{(2)}\phi_r^{(2)}\Big)\sin\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{0m}^{(2)}(k,\ell)\Bigg).$$

Let us see closer in each term. We start with Q_0^2. Thus we have

$$Q_0^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\ i_1^{(2)},\dots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m\beta_r\phi_r^{(1)}\Big)\cos\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{0m}^{(1)}(k,\ell)$$

$$-\sin\Big(\sum_{r=1}^m\beta_r^{(1)}\phi_r^{(1)}\Big)\cos\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{2m}^{(1)}(k,\ell)\Bigg)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m\beta_r^{(1)}\phi_r^{(1)}\Big)\sin\Big(\sum_{r=1}^m\beta_r^{(1)}\phi_r^{(1)}\Big)\sigma_{1m}^{(1)}(k,\ell)$$

$$-\sin\Big(\sum_{r=1}^m\beta_r^{(1)}\phi_r^{(1)}\Big)\sin\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{3m}^{(1)}(k,\ell)\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg)$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\sigma_{1m}^{(2)}(k,\ell)\right)$$

$$-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\Bigg)\Bigg).$$

Thus,

$$Q_0^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T \\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1 \\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\Bigg]$$

$$\times\Bigg[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg]$$

$$
\begin{aligned}
+ & \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \\
& \left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right] \\
& \times\left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\right] \\
- & \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \\
& \left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\right] \\
& \times\left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\right] \\
- & \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \\
& \left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(2)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right] \\
& \times\left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)\right. \\
& \left.-\sin\left(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\right].
\end{aligned}
$$

$$
Q_0^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T \\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1 \\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}
$$

$$
\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}
$$

$$
\Bigg[\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)
$$

$$
\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)+\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)
$$

$$
\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)
$$

$$
-\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)
$$

$$
\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)
$$

$$
-\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)
$$

$$
\cos\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\Bigg]
$$

$$
+\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}
$$

$$
\Bigg[\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)
$$

$$
\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)
$$

$$
+\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)
$$

$$\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)$$

$$\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$-\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)$$

$$\left.\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\right]$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\left[\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\right.$$

$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$-\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\left.\sin\left(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\right]$$

$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\left[\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\right)\right.$$

$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$
$$-\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$
$$-\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\Bigg]\Bigg).$$

We will do same procedure for 3 remains terms which we will find very similar expressions with variations of σ's.
For the second term we have

$$Q_1^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}}2^{1-m}$$
$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$
$$\left(\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\right.$$
$$\left.+\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)\right)$$
$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$
$$\left(\cos\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\right.$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)\Bigg)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\;\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)\Bigg)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\;\sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)\Bigg)\Bigg).$$

$$Q_1^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}}\;\sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N}\;\sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\;\sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$
$$+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\left.\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\right]$$
$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$
$$\left[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\right.$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$
$$+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\left.\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\right]$$
$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$
$$\left[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\right.$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$
\begin{aligned}
&+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\\
&+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\\
&+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\Bigg]\\
+&\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\;\sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\\
&\Bigg[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\\
&+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\\
&+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\\
&+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\Bigg]\Bigg).
\end{aligned}
$$

For the third term we have

$$Q_2^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m \beta_r\phi_r^{(1)}\Big)\cos\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{2m}^{(1)}(k,\ell)$$

$$+\sin\Big(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Big)\cos\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{0m}^{(1)}(k,\ell)\Bigg)$$

$$- \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}} \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Big)\sin\Big(\sum_{r=1}^m\beta_r^{(1)}\phi_r^{(1)}\Big)\sigma_{3m}^{(1)}(k,\ell)$$

$$+\sin\Big(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\Big)\sin\Big(\theta_1^{(1)}+\sum_{r=2}^m\alpha_r^{(1)}\theta_r^{(1)}\Big)\sigma_{1m}^{(1)}(k,\ell)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m \beta_r\phi_r^{(2)}\Big)\cos\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{2m}^{(2)}(k,\ell)$$

$$+\sin\Big(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Big)\cos\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{0m}^{(2)}(k,\ell)\Bigg)$$

$$- \sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}} \alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\Big(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Big)\sin\Big(\sum_{r=1}^m\beta_r^{(2)}\phi_r^{(2)}\Big)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sin\Big(\sum_{r=1}^m \beta_r^{(2)}\phi_r^{(2)}\Big)\sin\Big(\theta_1^{(2)}+\sum_{r=2}^m\alpha_r^{(2)}\theta_r^{(2)}\Big)\sigma_{1m}^{(2)}(k,\ell)\Bigg).$$

$$
\begin{aligned}
Q_2^2 = & \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\ i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m} \\
& \Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \\
& \left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\right. \\
& \quad \cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell) \\
& +\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right) \\
& \quad \cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell) \\
& +\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right) \\
& \quad \cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell) \\
& +\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right) \\
& \quad \left.\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\right] \\
& + \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} \\
& \left[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\right. \\
& \quad \sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)
\end{aligned}
$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$
$$+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\left.\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\right]$$
$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\;\sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$
$$\left[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\right.$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$
$$+\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\left.\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\right]$$
$$-\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\;\sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$
$$+\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$
$$+\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)$$
$$\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\Bigg]\Bigg).$$

For the fourth term we have

$$Q_3^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$
$$\Bigg[\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=+1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$
$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{3m}^{(1)}(k,\ell)$$
$$-\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{1m}^{(1)}(k,\ell)\Bigg)$$
$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\a_1^{(1)}\cdots a_m^{(1)}=-1}}\ \sum_{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}$$
$$\Bigg(\cos\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r^{(1)}\phi_r^{(1)}\right)\sigma_{2m}^{(1)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^{m}\beta_r^{(1)}\phi_r^{(1)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sigma_{0m}^{(1)}(k,\ell)\Bigg)$$

$$\times\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\ \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(2)}(k,\ell)\Bigg)$$

$$+\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(2)}\cdots a_m^{(2)}=-1}}\ \sum_{\alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg(\cos\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sigma_{2m}^{(2)}(k,\ell)$$

$$-\sin\left(\sum_{r=1}^{m}\beta_r^{(2)}\phi_r^{(2)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(2)}(k,\ell)\Bigg]. $$

Thus,

$$Q_3^2=\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ (k_1^{(2)},\ell_1^{(2)})\in T}}\ \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N}\ \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T\\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}$$

$$\Bigg(\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}}\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}$$

$$\Bigg[\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$+\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)$$

$$\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$
\begin{aligned}
&- \sin\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\cos\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \cos\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\\
&- \cos\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\cos\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \cos\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\Bigg]\\
+ &\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\\
&\Bigg[\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\sin\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\\
&+ \sin\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\sin\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\\
&- \sin\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\sin\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\\
&- \cos\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\sin\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\Bigg]\\
+ &\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\\
&\Bigg[\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r \phi_r^{(2)}\right)\cos\left(\theta_1^{(1)} + \sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\qquad \times \sin\left(\theta_1^{(2)} + \sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)
\end{aligned}
$$

$$
\begin{aligned}
&+\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\\
&-\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)\\
&-\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\sin\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)\Bigg]\\
+&\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}}\ \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\times\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\\
&\Bigg[\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\\
&+\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\\
&-\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)\\
&-\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\\
&\quad\times\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)\Bigg]\Bigg).
\end{aligned}
$$

Let us call the first sixteen common terms as following

$$
B_1^{(1)}=\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^m \beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^m \alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^m \alpha_r^{(2)}\theta_r^{(2)}\right)
$$

$$B_2^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_3^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_4^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$

$$B_5^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_6^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_7^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_8^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$

$$B_9^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{10}^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{11}^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{12}^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\cos\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\sin\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$

$$B_{13}^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{14}^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{15}^{(1)} = \sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right)$$
$$B_{16}^{(1)} = \cos\left(\sum_{r=1}^{m}\beta_r\phi_r^{(1)}\right)\sin\left(\sum_{r=1}^{m}\beta_r\phi_r^{(2)}\right)\sin\left(\theta_1^{(1)}+\sum_{r=2}^{m}\alpha_r^{(1)}\theta_r^{(1)}\right)\cos\left(\theta_1^{(2)}+\sum_{r=2}^{m}\alpha_r^{(2)}\theta_r^{(2)}\right).$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_1^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_1^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_2^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_2^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_3^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_3^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_4^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_4^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_5^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_5^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_6^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_6^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} \overset{A_7^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_7^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_8^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_8^{(1)} \sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_9^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_9^{(1)} \sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_{10}^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{10}^{(1)} \sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_{11}^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{11}^{(1)} \sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_{12}^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{12}^{(1)} \sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_{13}^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{13}^{(1)} \sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} \overset{A_{14}^{(1,2)}}{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{14}^{(1)} \sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}$$

$$A_{15}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{15}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{16}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)}^{A_{16}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{1}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}^{A_{17}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{2}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}^{A_{18}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{3}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}^{A_{19}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{4}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}^{A_{20}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{5}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}^{A_{21}^{(1,2)}}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \overbrace{\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{6}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)}^{A_{22}^{(1,2)}}$$

$$A_{23}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_7^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{24}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_8^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{25}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_9^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{26}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{10}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{27}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{11}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{28}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{12}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{29}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{13}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{30}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{14}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{31}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{15}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{32}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{16}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{33}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_1^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{34}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_2^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{35}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_3^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{36}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_4^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{37}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_5^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{38}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_6^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{39}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_7^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{40}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_8^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{41}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_9^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{42}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{10}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{43}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)} B_{11}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)$$

$$A_{44}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{12}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)$$

$$A_{45}^{(1,2)}$$

$$= \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\, B_{13}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}B_{14}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}_{A_{46}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{15}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)}_{A_{47}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=-1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{16}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}_{A_{48}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{1}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}_{A_{49}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{2}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}_{A_{50}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{3}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}_{A_{51}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=+1\\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{4}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}_{A_{52}^{(1,2)}}$$

$$A_{53}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_5^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{54}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_6^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{55}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_7^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{56}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}B_8^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{57}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_9^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$A_{58}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{10}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)$$

$$A_{59}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{11}^{(1)}\sigma_{1m}^{(1)}(k,\ell)\sigma_{2m}^{(2)}(k,\ell)$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{12}^{(1)}\sigma_{3m}^{(1)}(k,\ell)\sigma_{0m}^{(2)}(k,\ell)}_{A_{60}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{13}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}_{A_{61}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}\,B_{14}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}_{A_{62}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}B_{15}^{(1)}\sigma_{0m}^{(1)}(k,\ell)\sigma_{3m}^{(2)}(k,\ell)}_{A_{63}^{(1,2)}}$$

$$= \underbrace{\sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} \alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}B_{16}^{(1)}\sigma_{2m}^{(1)}(k,\ell)\sigma_{1m}^{(2)}(k,\ell)}_{A_{64}^{(1,2)}}.$$

We are now interested in to compute the real valued $|\left((HR_T)^m\sigma\right)_{k\ell}|^{2K} = (|\left((HR_T)^m\sigma\right)_{k\ell}|^2)^K = (|\left((HR_T)^m\sigma\right)_{k\ell}|^2)^K = (Q_0^2+Q_1^2+Q_2^2+Q_3^2)^K$. Hereby,

$$Q_0^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T \\ (k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1 \\ i_1^{(2)},\ldots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T \\ (k_2^{(2)},\ell_2^{(2)}),\ldots,(k_m^{(2)},\ell_m^{(2)})\in T \\ (k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)}) \\ (k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}\sum_{r=1}^{16} A_r^{(1,2)}$$

$$Q_1^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}\sum_{r=17}^{32} A_r^{(1,2)},$$

$$Q_2^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}\sum_{r=33}^{48} A_r^{(1,2)}$$

$$Q_3^2 = \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}\sum_{r=49}^{64} A_r^{(1,2)}.$$

Thus, we have

$$(Q_0^2+Q_1^2+Q_2^2+Q_3^2)^K$$

$$= \left(\sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\(k_1^{(2)},\ell_1^{(2)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\i_1^{(2)},\dots,i_m^{(2)}=1}}^{N} \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T\\(k_2^{(2)},\ell_2^{(2)}),\dots,(k_m^{(2)},\ell_m^{(2)})\in T\\(k_j^{(1)},\ell_j^{(1)})\neq(k_{j+1}^{(1)},\ell_{j+1}^{(1)})\\(k_j^{(2)},\ell_j^{(2)})\neq(k_{j+1}^{(2)},\ell_{j+1}^{(2)})}} 2^{1-m}\sum_{r=1}^{64} A_r^{(1,2)} \right)^K$$

$$= \sum_{\substack{(k_1^{(1)},\ell_1^{(1)})\in T\\ \vdots \\(k_1^{(K)},\ell_1^{(K)})\in T}} \sum_{\substack{i_1^{(1)},\dots,i_m^{(1)}=1\\ \vdots \\ \left(i_1^{(K)},j_1^{(K)}\right),\dots,\left(i_m^{(K)},j_m^{(K)}\right)=1}}^{N}$$

$$\times \sum_{\substack{(k_2^{(1)},\ell_2^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots \\ (k_2^{(K)},\ell_2^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{1-m}\prod_{p=1}^{K}\sum_{r=1}^{64}A_r^{(2p-1,2p)}$$

$$= \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ \vdots \\ \left(i_1^{(K)},j_1^{(K)}\right),\ldots,\left(i_m^{(K)},j_m^{(K)}\right)=1}}^{N} \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots \\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{1-m}$$

$$\times \sum_{(r_1,\ldots,r_p)\subset V(\{1,\ldots,64\},K)}\prod_{p=1}^{K}A_{r_p}^{(2p-1,2p)}$$

which $V(\{1,\ldots,64\},K)$ denotes the variations taking K elements from the set $\{1,\ldots,64\}$ with repetition.

For calculation the expected value $\mathbb{E}_X$ follows

$$\mathbb{E}_X\left[(Q_0^2+Q_1^2+Q_2^2+Q_3^2)^K\right] = \sum_{\substack{i_1^{(1)},\ldots,i_m^{(1)}=1\\ \vdots \\ \left(i_1^{(K)},j_1^{(K)}\right),\ldots,\left(i_m^{(K)},j_m^{(K)}\right)=1}}^{N}$$

$$\times \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots \\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{1-m} \sum_{(r_1,\ldots,r_K)\subset V(\{1,\ldots,64\},K)} \mathbb{E}_X\left[\prod_{p=1}^{K}A_{r_K}^{(2p-1,2p)}\right].$$

Taking in account that for each r, the we have $A_r^{(2p-1,2p)}$ in absolute value, the product of σ's for each term will vanishes. That means $|\sigma_{ij}^{(p)}(k,\ell)|=1$ on T and also $|\alpha_2^{\delta_2(1)}\cdots\alpha_m^{\delta_m(1)}\,\alpha_2^{\delta_2(2)}\cdots\alpha_m^{\delta_m(2)}|=1$. Thus

$$\mathbb{E}_X\left[(Q_0^2+Q_1^2+Q_2^2+Q_3^2)^K\right]\leq$$

$$\sum_{i_1^{(1)},\dots,i_m^{(1)}=1}^{N} \qquad \sum_{(k_1^{(1)},\ell_1^{(1)}),\dots,(k_m^{(1)},\ell_m^{(1)})\in T}$$
$$\vdots \qquad\qquad \vdots$$
$$\sum_{\left(i_1^{(K)},j_1^{(K)}\right),\dots,\left(i_m^{(K)},j_m^{(K)}\right)=1} \qquad \sum_{\substack{(k_1^{(K)},\ell_1^{(K)}),\dots,(k_m^{(K)},\ell_m^{(K)})\in T \\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}}$$
$$2^{1-m}\sum_{(r_1,\dots,r_K)\subset V(\{1,\dots,64\},K)}\mathbb{E}_X\left[\prod_{p=1}^{K}\widetilde{A}_{r_K}^{(2p-1,2p)}\right],$$

where the correspondent absolute value of $A_r^{(1,2)}$ denoted by $\widetilde{A}_r^{(1,2)}$ is given as following:

$$\widetilde{A}_1^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_1^{(1)}$$

$$\widetilde{A}_2^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_2^{(1)}$$

$$\widetilde{A}_3^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_3^{(1)}$$

$$\widetilde{A}_4^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_4^{(1)}$$

$$\widetilde{A}_5^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_5^{(1)}$$

$$\widetilde{A}_6^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\dots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\dots,\alpha_m^{(2)}\in\{-1,1\}}} B_6^{(1)}$$

$$\widetilde{A}_7^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_7^{(1)}$$

$$\widetilde{A}_8^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_8^{(1)}$$

$$\widetilde{A}_9^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_9^{(1)}$$

$$\widetilde{A}_{10}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{10}^{(1)})$$

$$\widetilde{A}_{11}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{11}^{(1)}$$

$$\widetilde{A}_{12}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{12}^{(1)}$$

$$\widetilde{A}_{13}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{13}^{(1)}$$

$$\widetilde{A}_{14}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{14}^{(1)}$$

$$\widetilde{A}_{15}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{15}^{(1)}$$

$$\widetilde{A}_{16}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{16}^{(1)}.$$

$$\widetilde{A}_{17}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{1}^{(1)}$$

$$\widetilde{A}_{18}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{2}^{(1)}$$

$$\widetilde{A}_{19}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{3}^{(1)}$$

$$\widetilde{A}_{20}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = +1 \\ a_1^{(2)}\cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{4}^{(1)}$$

$$\widetilde{A}_{21}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{5}^{(1)}$$

$$\widetilde{A}_{22}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)} = -1 \\ a_1^{(2)}\cdots a_m^{(2)} = -1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)} \in \{-1,1\}}} B_{6}^{(1)}$$

$$\widetilde{A}_{23}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_7^{(1)}$$

$$\widetilde{A}_{24}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_8^{(1)}$$

$$\widetilde{A}_{25}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_9^{(1)}$$

$$\widetilde{A}_{26}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{10}^{(1)}$$

$$\widetilde{A}_{27}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{11}^{(1)}$$

$$\widetilde{A}_{28}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{12}^{(1)}$$

$$\widetilde{A}_{29}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{13}^{(1)}$$

$$\widetilde{A}_{30}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{14}^{(1)}$$

$$\widetilde{A}_{31}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{15}^{(1)}$$

$$\widetilde{A}_{32}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{16}^{(1)}$$

$$\widetilde{A}_{33}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{1}^{(1)}$$

$$\widetilde{A}_{34}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{2}^{(1)}$$

$$\widetilde{A}_{35}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{3}^{(1)}$$

$$\widetilde{A}_{36}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{4}^{(1)}$$

$$\widetilde{A}_{37}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{5}^{(1)}$$

$$\widetilde{A}_{38}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{6}^{(1)}$$

$$\widetilde{A}_{39}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_7^{(1)}$$

$$\widetilde{A}_{40}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_8^{(1)}$$

$$\widetilde{A}_{41}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_9^{(1)}$$

$$\widetilde{A}_{42}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{10}^{(1)}$$

$$\widetilde{A}_{43}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{11}^{(1)}$$

$$\widetilde{A}_{44}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{12}^{(1)}$$

$$\widetilde{A}_{45}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{13}^{(1)}$$

$$\widetilde{A}_{46}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{14}^{(1)}$$

$$\widetilde{A}_{47}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{15}^{(1)}$$

$$\widetilde{A}_{48}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{16}^{(1)}$$

$$\widetilde{A}_{49}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{1}^{(1)}$$

$$\widetilde{A}_{50}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{2}^{(1)}$$

$$\widetilde{A}_{51}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{3}^{(1)}$$

$$\widetilde{A}_{52}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{4}^{(1)}$$

$$\widetilde{A}_{53}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{5}^{(1)}$$

$$\widetilde{A}_{54}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{6}^{(1)}$$

$$\widetilde{A}_{55}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_7^{(1)}$$

$$\widetilde{A}_{56}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_8^{(1)}$$

$$\widetilde{A}_{57}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_9^{(1)}$$

$$\widetilde{A}_{58}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{10}^{(1)}$$

$$\widetilde{A}_{59}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{11}^{(1)}$$

$$\widetilde{A}_{60}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=+1 \\ a_1^{(2)}\cdots a_m^{(2)}=-1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{12}^{(1)}$$

$$\widetilde{A}_{61}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{13}^{(1)}$$

$$\widetilde{A}_{62}^{(1,2)} = \sum_{\substack{\underline{a}\in\{-1,1\}^m \\ a_1^{(1)}\cdots a_m^{(1)}=-1 \\ a_1^{(2)}\cdots a_m^{(2)}=+1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\} \\ \alpha_2^{(2)},\ldots,\alpha_m^{(2)}\in\{-1,1\}}} B_{14}^{(1)}$$

$$\widetilde{A}_{63}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = -1 \\ a_1^{(2)} \cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)}, \ldots, \alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)}, \ldots, \alpha_m^{(2)} \in \{-1,1\}}} B_{15}^{(1)}$$

$$\widetilde{A}_{64}^{(1,2)} = \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = -1 \\ a_1^{(2)} \cdots a_m^{(2)} = +1}} \sum_{\substack{\alpha_2^{(1)}, \ldots, \alpha_m^{(1)} \in \{-1,1\} \\ \alpha_2^{(2)}, \ldots, \alpha_m^{(2)} \in \{-1,1\}}} B_{16}^{(1)}.$$

The relation between $\widetilde{A}_n^{(1,2)}$ and $B_n^{(1)}$ will be described as follows:

$$\begin{aligned} \widetilde{A}_n^{(1,2)} &\leftrightarrow B_n^{(1)}, \quad 1 \le n \le 16; \\ \widetilde{A}_n^{(1,2)} &\leftrightarrow B_{n-16}^{(1)}, \quad 16 < n \le 32; \\ \widetilde{A}_n^{(1,2)} &\leftrightarrow B_{n-32}^{(1)}, \quad 32 < n \le 48; \\ \widetilde{A}_n^{(1,2)} &\leftrightarrow B_{n-48}^{(1)}, \quad 48 < n \le 64. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbb{E}_X \left[(Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2)^K \right] \\ \le &\sum_{\substack{i_1^{(1)}, \ldots, i_m^{(1)} = 1 \\ \vdots \\ \left(i_1^{(K)}, j_1^{(K)}\right), \ldots, \left(i_m^{(K)}, j_m^{(K)}\right) = 1}}^{\tilde{N}} \sum_{\substack{(k_1^{(1)}, \ell_1^{(1)}), \ldots, (k_m^{(1)}, \ell_m^{(1)}) \in T \\ \vdots \\ (k_1^{(K)}, \ell_1^{(K)}), \ldots, (k_m^{(K)}, \ell_m^{(K)}) \in T \\ (k_j^{(p)}, \ell_j^{(p)}) \ne (k_{j+1}^{(p)}, \ell_{j+1}^{(p)})}} 2^2 \cdot 2^{1-m} \\ &\sum_{(r_1, \ldots, r_K) \subset V(\{1, \ldots, 16\}, K)} \mathbb{E}_X \left[\prod_{p=1}^K \widetilde{A}_{r_p}^{(2p-1,2p)} \left(\alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)}, \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} \right) \right]. \end{aligned}$$

B.1 Set partition

$$\begin{aligned} &\mathbb{E}_X \left[\prod_{p=1}^K \widetilde{A}_{r_p}^{(2p-1,2p)} \left(\alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)}, \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} \right) \right] \\ = &\mathbb{E}_X \left[\prod_{\{A, A'\} \in \mathcal{A}} \widetilde{A}_{r_p}^{(2p-1,2p)} \left(\alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)}, \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} \right) \right] \end{aligned}$$

$$= \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = \pm 1 \\ \vdots \\ a_1^{(K)} \cdots a_m^{(K)} = \pm 1}} \sum_{\substack{\alpha_2^{(1)}, \ldots, \alpha_m^{(1)} \in \{-1,1\} \\ \vdots \\ \alpha_2^{(K)}, \ldots, \alpha_m^{(K)} \in \{-1,1\}}} \prod_{\{A,A'\} \in \mathcal{A}}$$

$$\mathbb{E}_{X_A}\left[\lambda\left(\prod_{j=1}^m a_j^{(p)}, \sum_{(r,p)\in A} \alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)} x_A^{(p)}\right)\right] \mathbb{E}_{X_{A'}}\left[\lambda\left(\prod_{i=1}^m a_i^{(p)}, \sum_{(s,p)\in A'} \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} x_{A'}^{(p)}\right)\right].$$

$$\mathbb{E}_X\left[(Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2)^K\right]$$

$$\leq \sum_{\substack{i_1^{(1)}, \ldots, i_m^{(1)} = 1 \\ \vdots \\ \left(i_1^{(K)}, j_1^{(K)}\right), \ldots, \left(i_m^{(K)}, j_m^{(K)}\right) = 1}}^{\tilde{N}} \sum_{\substack{(k_1^{(1)}, \ell_1^{(1)}), \ldots, (k_m^{(1)}, \ell_m^{(1)}) \in T \\ \vdots \\ (k_1^{(K)}, \ell_1^{(K)}), \ldots, (k_m^{(K)}, \ell_m^{(K)}) \in T \\ (k_j^{(p)}, \ell_j^{(p)}) \neq (k_{j+1}^{(p)}, \ell_{j+1}^{(p)})}} 2^{3-m}$$

$$\sum_{(r_1,\ldots,r_K) \subset V(\{1,\ldots,16\},K)} \sum_{\substack{\underline{a} \in \{-1,1\}^m \\ a_1^{(1)} \cdots a_m^{(1)} = \pm 1 \\ \vdots \\ a_1^{(K)} \cdots a_m^{(K)} = \pm 1}} \sum_{\substack{\alpha_2^{(1)}, \ldots, \alpha_m^{(1)} \in \{-1,1\} \\ \vdots \\ \alpha_2^{(K)}, \ldots, \alpha_m^{(K)} \in \{-1,1\}}} \prod_{\{A,A'\} \in \mathcal{A}}$$

$$\mathbb{E}_{X_A}\left[\lambda\left(\prod_{j=1}^m a_j^{(p)}, \sum_{(r,p)\in A} \alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)} x_A^{(p)}\right)\right] \mathbb{E}_{X_{A'}}\left[\lambda\left(\prod_{i=1}^m a_i^{(p)}, \sum_{(s,p)\in A'} \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} x_{A'}^{(p)}\right)\right].$$

Let us pick up for now only the expected value. The variables (x_{i_r}, y_{i_r}) has uniform distribution on $[0, 2\pi]^2$, then follows

$$\prod_{\{A,A'\} \in \mathcal{A}} \mathbb{E}_{X_A}\left[\lambda\left(\prod_{j=1}^m a_j^{(p)}, \sum_{(r,p)\in A} \alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)} x_A^{(p)}\right)\right] \mathbb{E}_{X_{A'}}\left[\lambda\left(\prod_{i=1}^m a_i^{(p)}, \sum_{(s,p)\in A'} \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} x_{A'}^{(p)}\right)\right]$$
$$= \prod_{\{A,A'\} \in \mathcal{A}} \delta\left(\sum_{(r,p)\in A} \alpha_r^{(p)} \theta_r^{(p)} x_{i_r}^{(p)} x_A^{(p)}\right) \delta\left(\sum_{(s,p)\in A'} \beta_s^{(p)} \phi_s^{(p)} x_{j_s}^{(p)} x_{A'}^{(p)}\right).$$

Thus,

$$\mathbb{E}_X\left[|\left((HR_T)^m \sigma\right)_{k\ell}|^{2K}\right] =$$

$$
\begin{aligned}
&\leq \sum_{t=1}^{Km} \sum_{\mathcal{A}\in P(2?Km,t)} \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots \\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{3-m} \\
&\sum_{(r_1,\ldots,r_K)\subset V(\{1,\ldots,16\},K)} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=\pm1\\ \vdots\\ a_1^{(K)}\cdots a_m^{(K)}=\pm1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \vdots\\ \alpha_2^{(K)},\ldots,\alpha_m^{(K)}\in\{-1,1\}}} \prod_{\{A,A'\}\in\mathcal{A}} \\
&\prod_{\{A,A'\}\in\mathcal{A}} \delta\left(\sum_{(r,p)\in A} \alpha_r^{(p)}\theta_r^{(p)}x_{i_r}^{(p)}x_A^{(p)}\right)\delta\left(\sum_{(s,p)\in A'} \beta_s^{(p)}\phi_s^{(p)}x_{j_s}^{(p)}x_{A'}^{(p)}\right) \\
&\leq \sum_{t=1}^{\min\{Km,\tilde{N}\}} \frac{\tilde{N}!}{(\tilde{N}-t)!} \sum_{\mathcal{A}\in P(2?Km,t)} \sum_{\substack{(k_1^{(1)},\ell_1^{(1)}),\ldots,(k_m^{(1)},\ell_m^{(1)})\in T\\ \vdots \\ (k_1^{(K)},\ell_1^{(K)}),\ldots,(k_m^{(K)},\ell_m^{(K)})\in T\\ (k_j^{(p)},\ell_j^{(p)})\neq(k_{j+1}^{(p)},\ell_{j+1}^{(p)})}} 2^{3-m} \\
&\sum_{(r_1,\ldots,r_K)\subset V(\{1,\ldots,16\},K)} \sum_{\substack{\underline{a}\in\{-1,1\}^m\\ a_1^{(1)}\cdots a_m^{(1)}=\pm1\\ \vdots\\ a_1^{(K)}\cdots a_m^{(K)}=\pm1}} \sum_{\substack{\alpha_2^{(1)},\ldots,\alpha_m^{(1)}\in\{-1,1\}\\ \vdots\\ \alpha_2^{(K)},\ldots,\alpha_m^{(K)}\in\{-1,1\}}} \prod_{\{A,A'\}\in\mathcal{A}} \\
&\prod_{\{A,A'\}\in\mathcal{A}} \delta\left(\sum_{(r,p)\in A} \alpha_r^{(p)}\theta_r^{(p)}x_{i_r}^{(p)}x_A^{(p)}\right)\delta\left(\sum_{(s,p)\in A'} \beta_s^{(p)}\phi_s^{(p)}x_{j_s}^{(p)}x_{A'}^{(p)}\right),
\end{aligned}
$$

where $\theta_r^{(p)} = \left(k_{r+1}^{(p)} - k_r^{(p)}\right)$ and $\phi_s^{(p)} = \begin{cases} (\ell_{s+1}^{(p)} - \ell_s^{(p)}), & a_s = +1 \\ -(\ell_{s+1}^{(p)} + \ell_s^{(p)}), & a_s = -1, \end{cases}$ which $\alpha_r^{(p)} \in \{-1,1\}$ and $\beta_s^{(p)} = (-1)^{\pi(s)}$ such that $\pi(s) = \sum_{j=1}^{s-1} \frac{|a_j|-a_j}{2}$ which counts the number of $\sin$'s.

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